Noncommutative localization
in noncommutative geometry

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Abstract

The aim of these notes is to collect and motivate the basic localization toolbox for the geometric study of "spaces" locally described by noncommutative rings and their categories of modules.

We present the basics of ORE localization of rings and modules in great detail. Common practical techniques are studied as well. We also describe a counterexample to a folklore test principle for Ore sets. Localization in negatively filtered rings arising in deformation theory is presented. A new notion of the differential ORE condition is introduced in the study of localization of differential calculi.

To aid the geometrical viewpoint, localization is studied with emphasis on descent formalism, flatness, abelian categories of quasicoherent sheaves and generalizations, and natural pairs of adjoint functors for sheaf and module categories. The key motivational theorems from the seminal works of GABRIEL on localization, abelian categories and schemes are quoted without proof, as well as the related statements of POPESCU, EILENBERG-WATTS, DELIGNE and ROSENBERG.

The COHN universal localization does not have good flatness properties, but it is determined by the localization map already at the ring level, like the perfect localizations are. Cohn localization is here related to the quasideterminants of GELFAND and RETAKH; and this may help the understanding of both subjects.

Contents

1 Introduction

1.1 Objectives and scope. This is an introduction to Ore localizations and generalizations, with *geometric* applications in mind.

The existing bibliography in localization theory is relatively vast, including several monographs. Localizations proliferate particularly in category theory, with flavours adapted to various situations, like bicategories, toposes, QUILLEN's model categories, triangulated categories etc. A noncommutative algebraic geometer replaces a space with a ring or more general 'algebra', or with some category whose objects mimic modules over the 'algebra', or mimic sheaves over a space, or he/she studies a more general category which is glued from such local ingredients. This setup suggests that we may eventually need a similar toolbox to the one used by the category and homotopy

theorists; however the simplest cases are in the area of ring and module theory. Even so, we shift the emphasis from purely ring-theoretic questions to more geometrical ones.

A **localized ring** is typically structurally simpler than the original, but retaining some of its features. A useful tool for a ring theorist is a controlled but *substantial simplification* of a ring, often as extreme as passing to a local ring or a quotient skewfield. On the contrary, our main geometrical goal are *those localizations which may play the role of noncommutative analogues of (rings of regular functions on principal Zariski) open sets* in an (affine, say) variety. Rings of functions on these open sets may be slightly simpler than the rings of *global* functions, but not as radically as when, say, passing to a local ring. We start with the very basics of localization procedures. The geometric notion of a *cover* by localizations is studied in the noncommutative context as well. Only recent geometrically minded works include some elements of such study, as some key features of covers, the *globalization lemma* in particular, were recognized only in mid-eighties.

We use an *elementary method* to prove the existence and simple properties of the Ore localized rings, in line with the original 1931 paper of O. ORE [98] (who however assumed no zero divisors). Modern treatments often neglect details when sketching the elementary method. Another modern method (to prove existence etc. ([91])), following ASANO ([6]), is cheap, but does not give an equivalent experience with the Ore method. Calculations similar to the ones in our proofs appear in concrete examples when using or checking Ore conditions *in practice*. We also use this method to examine when there is an induced localization of a first order differential calculus over a noncommutative ring, and come to a condition, not previously observed, which we call the "differential Ore condition". The elementary method has the advantage of being parallel to the calculus of (left) fractions in general categories, which has an aura of being a difficult subject, but is more transparentafter learning Ore localization the elementary way.

Our next expositional goal is to obtain some practical criteria for finding and dealing with Ore localizations in practice. Folklore strategies to prove that some set is Ore are examined in detail. In a section on Ore localization in 'negatively' filtered rings, we explore similar methods.

While Ore localization is treated in a comprehensive introductory style, more general localizations are sketched in a survey style. For advanced topics there are often in place good references, assuming that the reader knows the motivation, and at least the Ore case. Both requisites may be fulfilled by reading the present notes. We emphasize facts important in

geometry, often omitted, or which are only folklore. In order to clear up some sources of confusion, we sketch and compare several levels of generality; mention competing terminologies; and examine difficulties of geometrical interpretation and usage.

We focus on localizations of the category $R - \mathrm{Mod}$ of all *left* modules over a fixed ring R. Localization in other specific categories, e.g. central bimodules (symmetric localization, cf. [62]), bimodules, and the standard approach via *injective hulls* are omitted. One of the reasons is that often there are too few central bimodules over a noncommutative ring and 2-sided ideals in particular. Bimodules in general are interpreted as generalizing the maps of noncommutative rings, as explained in the text. Generalities on localization in arbitrary categories, and abelian in particular, are outlined for convenience.

As COHN localization can be found in other works in this volume, we include only a short introduction with two goals: putting it in our context and, more importantly, relating it to the recent subject of quasideterminants. Anybody aware of both subjects is aware of some connection, and we try to spell it out as precisely as we can.

1.2 Prerequisites on algebraic structures. Basic notions on rings and modules are freely used: unital ring, left ideal, center, left module, bimodule, domain (ring with no zero divisors), skewfield (division ring), graded rings and modules, and operations with those.

1.3 Prerequisites on categories. The language of categories and functors is assumed, including a (universal) initial and terminal object, (projective=inverse) limit, colimit (= inductive/direct limit), (co)products, adjoint functors, Yoneda lemma, and the categorical duality (inverting arrows, dual category, dual statements about categories).

Appendix A in [144] suffices for our purposes; for more see [23, 21, 84].

1.4 A morphism $f : B \to C$ is epi (mono) if for any pair e, e' of morphisms from C to any D (resp. from any A to B), equality $ef = e'f$ (resp. $fe = fe'$) implies $e = e'$.

A **subobject** is an equivalence class of monomorphisms. A pair (F, in) consisting of a functor F, and a natural transformation of functors in : $F \hookrightarrow G$, is a **subfunctor** of functor G, if all $\mathrm{in}_M : F(M) \hookrightarrow G(M)$ are monomorphisms. Explicitly, the naturality of in reads $\forall f : M \to N$, $\mathrm{in}_N \circ F(f) = G(f) \circ \mathrm{in}_M : F(M) \to G(N)$. Clearly, if F is a subfunctor of an additive (**k**-linear) functor G, between additive (**k**-linear) categories, then F is additive (**k**-linear) as well.

1.5 A (small) diagram d in the category $\mathcal{C}$ will be viewed as a functor

from some (small) category D into $\mathcal{C}$. For fixed D, such functors and natural transformations make a category $\mathcal{C}^D$. Every object c in $\mathcal{C}$ gives rise to a constant diagram c^D sending each X in D into c. This rule extends to a functor $(\ )^D : \mathcal{C} \to \mathcal{C}^D$. A **cone over diagram** $d : D \to \mathcal{C}$ is a natural transformation $I_c : c^D \Rightarrow d$ for some $c \in \mathcal{C}$. A morphism $I_c \to I_{c'}$ of cones over d is a morphism $\phi : c' \to c$ such that $I_{c'} = I_c \circ \phi^D$. A terminal among the cones over D will be called a **limiting cone** over D. A **colimiting cone** in $\mathcal{C}$ is a limiting cone in opposite category $\mathcal{C}^{\mathrm{op}}$. Consider a 'parallel' family of morphisms $\{f_\gamma : A \to B\}_{\gamma \in \Gamma}$ as a diagram D with 2 objects and $|\Gamma|$ arrows in an obvious way. In this case, a cone over D is given by a single map $I : c \to A$. We call the diagram $c \xrightarrow{I} A \overset{f_\gamma}{\Rightarrow} B$ a **fork diagram**. It is called an **equalizer (diagram)** if $I : c \to A$ is in addition a limiting cone; by abuse of language one often calls I, or even c an equalizer. Equalizers in $\mathcal{C}^{\mathrm{op}}$ are referred to as **coequalizers**. A morphism $I \to A$ of the cone of an equalizer diagram with $\Gamma = \{1, 2\}$ is also called a **kernel** of parallel pair $f_1, f_2 : A \to B$. Cokernels are defined dually.

A **zero** object 0 is an object which is simultaneously initial and terminal. In that case, composition $X \to 0 \to X$ is also denoted by $0 : X \to X$. A (co)kernel of a *single* morphism $f : A \to B$ in a category with 0 is by definition a (co)kernel of pair $f, 0 : A \to B$.

1.6 A functor $F : \mathcal{C} \to \mathcal{C}'$ induces a (pullback) functor for diagrams $F \circ : \mathcal{C}^D \to (\mathcal{C}')^D$. It is defined by $d \mapsto F \circ d$ for every diagram $d : D \to \mathcal{C}$, and $\alpha \mapsto F(\alpha)$ where $(F(\alpha))_M := F(\alpha_M)$ for $\alpha : d \Rightarrow d'$.

F **preserves** limits of some class $\mathcal{P}$ of diagrams in $\mathcal{A}$ if it sends any limiting cone $p_0 \to p$ over any $p \in \mathcal{P}$ in $\mathcal{A}$ into a limiting cone in $\mathcal{A}'$. F **reflects** limits if any cone $p_0 \to p$ over any $p \in \mathcal{P}$ in $\mathcal{A}$ must be a limiting cone if F sends it to a limiting cone in $\mathcal{A}'$. The same holds whenever the word 'limit' is replaced by 'colimit', and cone $p_0 \to p$ by a cocone $p \to p_0$.

1.7 An **Ab**-category (or **preadditive category**) is a category $\mathcal{A}$ with an abelian group operation $+$ on each set $\mathcal{A}(X, Y)$, such that each composition map $\circ : \mathcal{A}(X, Y) \times \mathcal{A}(Y, Z) \to \mathcal{A}(Y, Z)$ is bilinear. An **Ab**-category is **additive** if it contains a zero object and pairwise, hence all finite, products. Automatically then, finite products agree with finite coproducts. Recall that an additive category $\mathcal{A}$ is **abelian** if each morphism f in $\mathcal{A}$ has a kernel and a cokernel morphism, and the kernel object of a cokernel equals the cokernel object of a kernel. We assume that the reader is comfortable with elementary notions on abelian groups like exact sequences and left (right) exact functors in the greater generality of abelian categories.

1.8 Gabriel-Mitchell-Popescu embedding theorem. *Every small*

abelian category is equivalent as an abelian category to a subcategory of category of left modules over a certain ring R. Proof: [103].

1.9 Prerequisites on spaces of modern geometry. We expect familiarity with the notions of a *presheaf, separated presheaf* and *sheaf,* and with some examples describing a geometry via a topological space with a structure sheaf on it; as well as the idea of gluing from some sort of local models, behind the concepts of (super)manifolds, analytic spaces and schemes. Earlier exposure to commutative algebraic (or analytic) varieties and schemes is assumed in style of some sections, but no specific facts are required; an abstract sketch of the main features of scheme-like theories is supplied in the text below.

1.10 Conventions. The word *map* means a set-theoretic map unless it is accompanied with a specification of the ambient category in question when it means a (homo)morphism, e.g. a *map of rings* means a ring (homo)morphism. The word *noncommutative* means "not necessarily commutative". Though for many constructions this is not necessary, we mostly deal with unital rings and modules, unless said otherwise. *Ideal* without a modifier always means '1-sided (usually left) ideal'.

The symbol for inclusion $\subset$ may include the case of equality. The unadorned tensor symbol is over $\mathbb{Z}$, except for elements in given tensor products, like $a \otimes b := a \otimes_R b \in A \otimes_R B$. For algebras and modules over a commutative ring, this ring is usually denoted $\mathbf{k}$. These conventions may be locally overwritten by contextual remarks.

2　Noncommutative geometry

DESCARTES introduced the method of coordinates, which amounted to the identification of real vector spaces with the spaces described by the axioms of EUCLID. LAGRANGE considered more general curvilinear coordinates in analytic mechanics to obtain exhaustive treatments of space. Topological spaces do not have distinguished coordinate functions, but the whole algebra of functions suffices. The GELFAND-NEIMARK theorem (e.g. [74]) states that the category of compact Hausdorff topological spaces is equivalent to the opposite of category of commutative unital C^*-algebras. This is accomplished by assigning to a compact X the Banach $\star$-algebra $C(X)$ of all continuous $\mathbb{C}$-valued functions (with the supremum-norm and involution $f^*(x) = \overline{f(x)}$). In the other direction one (re)constructs X as a Gelfand spectrum of the algebra A, which is a space whose points are continuous characters $\chi : A \to \mathbb{C}^*$ endowed with spectral topology. These characters appear as the evaluation

functionals χ_x on A at points $x \in X$, where $\chi_x(f) = f(x)$. Each annihilator $\operatorname{Ann} \chi = \{ a \in A \,|\, \chi(a) = 0 \}$ is a maximal ideal of A and all maximal ideals are of that form.

2.1 Towards noncommutative algebraic geometry. For any commutative ring R, GROTHENDIECK replaced maximal ideals from the theory of affine varieties and from the Gelfand-Neimark picture, by arbitrary prime ideals, which behave better functorially, and he endowed the resulting spectrum $\operatorname{Spec} R$ with a non-Hausdorff Zariski topology and a structure sheaf, defined with the help of commutative localization. This amounts to a fully faithful contravariant functor Spec from $\underline{CommRings}$ to the category $\mathbb{lSp}$ of locally ringed spaces. In other words, the essential image of this functor, the category of geometric affine schemes $\text{Aff} = \operatorname{Spec}(\underline{CommRings})$ is equivalent to the formal category of affine schemes which is by the definition $\underline{CommRings}^{\mathrm{op}}$. Notions of points, open subspaces and sheaves are used to define $\mathbb{lSp}$ and the functor Spec. The functor Spec takes values in a category described in local geometrical terms, translating algebraic concepts into geometric ones. The functor enables the transfer of intuition and methods between algebra and geometry. This interplay is to a large extent the basic *raison d'être* for the subject of algebraic geometry. The spaces in $\mathbb{lSp}$ may be glued via topologies and sheaves, and certain limit constructions may be performed there what gives a great flexibility in usage of range of other subcategories in $\mathbb{lSp}$ e.g. schemes, algebraic spaces, formal schemes, almost schemes etc. Useful constructions like blow-ups, quotients by actions of groups, flat families, infinitesimal neighborhoods etc. take us often out of the realm of affine schemes.

The dictionary between the geometric properties and abstract algebra may be partially extended to include noncommutative algebra. Noncommutative geometry means exploring the idea of faithfully extending the Spec functor (or, analogously, the Gelfand's functor above) to noncommutative algebras as a domain and some geometrical universe $\mathfrak{nlSp}$ generalizing the category $\mathbb{lSp}$ as a target, and thinking geometrically of the consequences of such construction(s). The category $\mathfrak{nlSp}$ should ideally contain more general noncommutative schemes, extending the fact that spaces in $\mathbb{lSp}$ feature topology, enabling us to glue affine schemes over such subsets. Useful examples of noncommutative spaces are often studied in contexts which are much more restricted then what we require for $\mathfrak{nlSp}$; for instance one gives away topology or points, or one works only with Noetherian algebras close to commutative (say of finite GK-dimension, or finite dimensional over the center, when the latter is a field), or the category is not big enough to include the

whole of $\mathbb{l}\mathfrak{S}\mathfrak{p}$. For example, VAN OYSTAEYEN and his school ([138, 139, 141]) consider a certain class of graded rings for which they can use localizations to define their version of noncommutative **Proj**-functor. A more restricted class of graded rings supplying examples very close in behaviour to commutative projective varieties is studied by ARTIN, ZHANG, STAFFORD and others (see [5, 127] and refs. therein). Y. SOIBELMAN ([126]) advocates examples of natural compactifications of moduli spaces of commutative spaces with noncommutative spaces as points on the boundary.

Thus we often restrict ourselves either to smaller geometric realms Ns $\subset$ $\mathfrak{n}\mathbb{l}\mathfrak{S}\mathfrak{p}$ than $\mathfrak{n}\mathbb{l}\mathfrak{S}\mathfrak{p}$ containing for example only projective "noncommutative varieties" of some sort, or to give up points, topological spaces in ordinary sense and work with a more intrinsic embedding of the category of affine schemes (now $Rings^{\mathrm{op}}$ into some category of (pre)sheaves over $Rings^{\mathrm{op}}$ using Yoneda lemma, Grothendieck topologies and related concepts ([69, 99, 110]). In the commutative case, while both the spectral and functorial approaches are interchangeably used (EGA prefers spectral, while SGA and DEMAZURE-GABRIEL [31] choose functorial; the latter motivated with niceties in the treatment of group schemes), the more difficult foundational constructions of theoretical nature are done using the Yoneda embedding approach (algebraic geometry over model categories ([132, 133]); THOMASON's work ([134]) on K-theory and derived categories (cf. also [7]); $\mathbb{A}^1$-homotopy theory of schemes ([92])).

2.2 One often stops consciously half way toward the construction of the functor $Rings^{\mathrm{op}} \rightarrow \mathfrak{n}\mathbb{l}\mathfrak{S}\mathfrak{p}$; e.g. start with rings and do nothing except for introducing a small class of open sets, e.g. commutative localizations, ignoring other natural candidates, because *it is difficult to work with them*. Unlike in the above discussed case of restricting the class of spaces Ns $\subset$ $\mathfrak{n}\mathbb{l}\mathfrak{S}\mathfrak{p}$, we are conservative in the details of the spectral description, thus landing in some intermediate "semilocal" category **slSp** by means of a fully faithful embedding $Rings^{\mathrm{op}} \rightarrow$ **slSp**.

2.2a An example of **slSp** is as follows. Consider the center $Z(R)$ of a ring R and construct the commutative ring $\mathrm{Spec}\, Z(R)$. Then for each principal open set U in $Z(R)$, one localizes R at U (a commutative localization) and this essentially gives the structure sheaf $U \rightarrow R_U = \mathcal{O}(U)$. The problem is, what if the center is small, hence $\mathrm{Spec}\, Z(R)$ is small as well, hence all the information on R is kept in a few, maybe one, rings R_U, and we did not get far. In some cases the base space $\mathrm{Spec}\, Z'$ is big enough and we may glue such spectra to interesting more general "schemes" ([106]). Taking the center is not functorial, so we have to modify the categories a bit, to allow

for pairs (R, C), $C \subset Z(R)$ from the start, and construct from them some "space" $\mathrm{Spec}_2(R, C)$ (see more in Ch. 10). It is argued in [106] that, when R is small relative to Z', this construction is a satisfactory geometrization and the standard tools from cohomology theory may be used. They call such a situation *semiquantum geometry.*

A fruitful method is to add a *limited class* $\{Q_i\}_i$ of other localizations on $R - \mathrm{Mod}$, and *think* of $Q_i R$ as the structure ring R_{U_i} over open subset U_i. However, now U_i is not really a subset in $\mathrm{Spec}\, Z(R)$, but rather a "geometric" label for Q_i viewed as certain open set on hypothetical noncommutative $\mathrm{Spec}\, R$. Of course the latter point of view is central ([69, 62, 111, 137, 138, 141]) for our subject far beyond the idea of a small enrichment of the Zariski topology on the spectrum of the center.

In summary, restricting sharply to a small class of localizations and/or working with small spectra, projects a coarser local description $\mathbf{slSp}(R)$ than often desired.

2.3 Alternatively, one may loose some information, for instance considering the points of spectra but not the sheaves, or types of spectra with insufficiently many points for the reconstruction of rings. We may think of such correspondences as nonfaithful functors from $\mathbf{slSp}$ into some partial geometric realms $\mathrm{Feature}_\alpha(\mathfrak{nlGp})$.

2.4 MANIN has suggested ([89]) a functor from graded rings into abelian categories: to a Noetherian ring R assign the quotient of the category of finitely generated graded R-modules by the subcategory of the finitely generated graded R-modules of finite length. In the commutative case, by a theorem of SERRE, this category is equivalent to the category of *coherent sheaves* over $\mathbf{Proj}\, R$. This is one of the candidates for *projective noncommutative geometry* and we view it as an example of functor of type $\mathbf{slSp}$. Manin here actualizes the GROTHENDIECK's advice that *to do geometry one does not need the space itself but only the category of sheaves on that would-be space.* In this spirit, Grothendieck defined **topos** ([63] and [21], vol.3) as an abstract category satisfying a list of axioms, whose consequence is that it is equivalent to the category of sheaves $\mathrm{Fas}\,\mathcal{C}$ over some site $\mathcal{C}$ (a category with a Grothendieck topology). Two different sites may give rise to the same topos, but their cohomological behaviour will be the same. Thus they are thought of as the same generalized space. Likewise, in algebraic geometry, we have examples for the same heuristics, where abelian categories of quasi-coherent sheaves of $\mathcal{O}$-modules, are in the place of the topos of *all* sheaves of sets. The suitable notion of a morphism between the topoi is recognized to be 'geometrical morphism' what is also an adjoint pair of functors with cer-

tain additional properties. In topos theory and applications, Grothendieck actually utilizes an interplay ('yoga') of 6 *standard* functors attached to the same 'morphism'. We shall discuss the basic pairs of adjoint functors for the categories of modules and sheaves. They appear in the disguise of maps of (noncommutative) rings (affine maps and their abstract version), as bimodules for two rings, as direct and inverse maps for $\mathcal{O}_X$-modules, and as localization functors.

2.5 Grothendieck categories (G.c.) [62, 125]. A Grothendieck category is a cocomplete (having all small limits) abelian category, having enough injectives and a small generator. The category of left R-modules, and the category of all sheaves of left R-modules over a fixed topological space, are G.c.'s. Given a coalgebra C, the category of C-comodules is G.c. Given a bialgebra B and a B-comodule algebra E, the category of relative (E, B)-Hopf modules is a G. c. [145].

Theorem. (P. GABRIEL, [42]) *The category $\mathfrak{Qcoh}_X$ of quasicoherent sheaves of $\mathcal{O}_X$-modules over a quasicompact quasiseparated scheme X is a Grothendieck category.*

It is *not* known if $\mathfrak{Qcoh}_X$ where X is a general scheme is cocomplete nor if it has enough injectives. This fact is behind our decision not to strictly require our abelian categories of noncommutative geometry to be a G.c. (which is fashionable). ROSENBERG [109] requires the weaker *property (sup)* (= *categories with exact limits*) introduced by GABRIEL ([42]): for any object M and any ascending chain of subobjects there is a supremum subobject, and taking such supremums commutes with taking the join (minimum) with a fixed subobject $N \subset M$. This holds for R-mod, Fas $\mathcal{C}$ (for a small site $\mathcal{C}$) and $\mathfrak{Qcoh}_X$ (for any scheme X).

2.6 Theorem. (P. GABRIEL for noetherian schemes ([42], Ch. VI); A.L. ROSENBERG in quasicompact case ([108]); and in general case ([115]))
Every scheme X can be reconstructed from the abelian category $\mathfrak{Qcoh}_X$ uniquely up to an isomorphism of schemes.

This motivates the promotion to a "space" of any member of a class of abelian categories, usually required to obey some additional axioms, permitting for (some variant) of R-mod (R possibly noncommutative) and $\mathfrak{Qcoh}_X$ as prime examples. A distinguished object $\mathcal{O}$ in $\mathcal{A}$, corresponding to the structure sheaf is often useful part of a data, even at the abstract level, hence the spaces could be actually pairs $(\mathcal{A}, \mathcal{O})$. The study of functors for the categories of modules and categories of sheaves shows that there is a special role for functors having various exactness properties ([109]), e.g. having a right adjoint, hence such properties are often required in general. Gluing cate-

gories over localizations, and variants thereof, should be interpreted in good cases as gluings of spaces from local models. In noncommutative geometry, a local model is usually the full category of modules over a noncommutative ring.

2.7 The so-called *derived algebraic geometry*, treating in more natural terms and globalizing the infinitesimal picture of moduli spaces governed by the deformation theory, appeared recently ([12, 132]). Its cousin, *homotopical algebraic geometry* appeared promising in the study of homotopy theories for algebraic varieties, and also in using the reasoning of algebraic geometry for ring spectra of homotopy theory and for their globalization. In such generalizations of algebraic geometry the basic gadgets are *higher categories* (e.g. simplicially enriched, DG, Segal, A_∞, cf. [34, 65, 105, 132, 133]). The lack of smoothness in some examples of moduli spaces is now explained as an artifact of the truncation process replacing the natural and smooth 'derived moduli spaces' by ordinary moduli spaces ('*hidden smoothness principle*' due to BONDAL, DELIGNE, DRINFELD, KAPRANOV, HINICH, KONTSEVICH...).

Part of the relevant structure here may be already expressed by replacing rings by differential graded algebras (dga-s) ([12]), or, more generally, by introducing sequences of higher (e.g. 'MASSEY') products, as in the theory of A_∞ (strongly homotopy associative) and L_∞ (strongly homotopy Lie) algebras. Such generalizations and special requirements needed to do localization in such enriched settings, are beyond the scope of the present article. A noncommutative algebraic geometry framework designed by O.A. LAUDAL ([75]), with emphasis on the problem of noncommutative deformation of moduli, implicitly includes the higher Massey products as well. In the viewpoint put forward by KONTSEVICH and FUKAYA, some of the 'dualities' of modern mathematical physics, e.g. the *homological mirror symmetry*, involve A_∞-categories defined in terms of geometric data ([70, 88, 126]). The so-called **quantization** ([30, 67, 74, 147]) in its many versions is generally of deformational and noncommutative nature. Thus it is not surprising that the formalisms combining the noncommutative and homological (or even homotopical) structures benefit from the geometrically sound models of quantum physics. MANIN suggested that a more systematic content of a similar nature exists, programmatically named *quantized algebraic geometry*, which may shed light on hidden aspects of the geometry of (commutative) varieties, including the deep subject of motives.

An interesting interplay of derived categories of coherent sheaves on varieties and their close analogues among other triangulated categories, motivates some 'noncommutative' geometry of the latter ([20, 19]). Triangulated

categories are also only a "truncation" of some other higher categories.

One should also mention that some important classes of rings in *quantum algebra*, for example quantum groups, may be constructed using categories of (perverse) sheaves over certain commutative configuration spaces ([83]). Thus the structure of various sheaf categories is an ever repeating theme which relates the commutative and noncommutative world. See the essay [26] for further motivation.

3   Abstract localization

We discuss localization of 1. algebraic structures; 2. categories. These two types are related: typically a localization of a ring R induces a localization of the category $R - \mathrm{Mod}$ of left modules over R.

A recipe $\mathcal{G}$ for a localization takes as input a structure R (monoid, lattice, ring), or a category $\mathcal{A}$, and a distinguished data Σ in R (or $\mathcal{A}$). The localizing data Σ is selected from some class $\mathcal{U}(R)$ of structural data, for example elements, endomorphisms or ideals of R; similarly $\mathcal{U}(\mathcal{A})$ could be a class of subcategories or collection of morphisms in $\mathcal{A}$. Usually not all obvious subclasses of $\mathcal{U}(R)$ may serve as distinguished data for $\mathcal{G}$, and some 'localizability' conditions apply.

A localization procedure $\mathcal{G}(R, \Sigma)$ should replace R by another object Y and a map $i : R \to Y$, which induces, for given $\mathcal{G}$, some canonical correspondence $\mathcal{G}(i) : \Sigma \rightsquigarrow \Sigma_*$ between the localization data Σ and some other data Σ_* chosen from $\mathcal{U}(Y)$. The subclass Σ_* should satisfy some natural requirement, for example that it consist of invertible elements. Pair (i, Y) should be in some sense smallest, or universal among all candidates satisfying the given requirements. For given requirements only certain collections Σ built from elements in $\mathcal{U}(R)$ give rise to a universal (i, Y). Such Σ are generically called *localizable* and the resulting Y is denoted $\Sigma^{-1}R$.

In the case of a category $\mathcal{C}$, a map i is replaced by a localization functor $Q^* : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$. In this article, a localization of a category will be equivalent to an abstract localization with respect to a class of morphisms Σ in $\mathcal{C}$, often using some other equivalent data (e.g. 'localizing subcategory'). Following [43], we sketch the general case of a localization at a class of morphisms Σ, cf. also [21].

3.1 An **abstract 1-diagram** $\mathcal{E}$ is a structure weaker then a category: it consists of a class $\mathrm{Ob}\,\mathcal{E}$ of objects and a class $\mathrm{Mor}\,\mathcal{E}$ of morphisms equipped with a source and a target maps $\mathrm{s}, \mathrm{T} : \mathrm{Mor}\,\mathcal{E} \to \mathrm{Ob}\,\mathcal{E}$. No composition, or identity morphisms are supplied. As usual, for two objects A, B by $\mathcal{E}(A, B)$

we denote class of morphisms f with $\mathrm{s}(f) = A$ and $\mathrm{T}(f) = B$. If each $\mathcal{E}(A, B)$ is a set, one may use word (multiple-edge) graph instead. If $\mathcal{E}, \mathcal{C}$ are diagrams, an $\mathcal{E}$-**diagram** in $\mathcal{C}$, or a morphism $d : \mathcal{E} \to \mathcal{C}$, is any pair of maps $\mathrm{Ob}\,\mathcal{E} \to \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{E} \to \mathrm{Mor}\,\mathcal{C}$ which commute with source and target maps. Small abstract 1-diagrams and their morphisms form a category $\mathfrak{Diagr}_1$. To each category one assigns its underlying abstract diagram. This correspondence induces a forgetful functor from the category $\mathfrak{Cat}$ of small categories to $\mathfrak{Diagr}_1$. The construction of a category of paths below provides the left adjoint to this functor.

If $n \geq 0$ is an integer, a **path** of length n from A to B in an abstract diagram $\mathcal{E}$ is a tuple $(A, f_1, f_2, \ldots, f_n, B)$, where A is an object and f_i are morphisms in $\mathcal{E}$, such that $\mathrm{s}(f_{i+1}) = \mathrm{T}(f_i)$ for $i = 1, \ldots, n-1$, and $\mathrm{s}(f_1) = A$, $\mathrm{T}(f_n) = B$ if $n > 0$, and $A = B$ if $n = 0$. For any abstract 1-diagram $\mathcal{E}$ define a **category** $\mathrm{Pa}\,\mathcal{E}$ **of paths** in $\mathcal{E}$ as follows. The class of objects is

$$\mathrm{Ob}\,\mathrm{Pa}\,\mathcal{E} := \mathrm{Ob}\,\mathcal{E}$$

and the morphism class $(\mathrm{Pa}\,\mathcal{E})(A, B)$ consists of all paths from A to B. One declares $\mathrm{Id}_A := (A, A)$, $\mathrm{s}'(A, f_1, \ldots, f_n, B) = A$ and $\mathrm{T}'(A, f_1, \ldots, f_n, B) = B$ to be the identity morphisms, and the source and target maps for $\mathrm{Pa}\,\mathcal{E}(A, B)$. The composition rule is

$$(A, f_1, \ldots, f_n, B) \circ (B, g_1, \ldots, g_m, C) = (A, f_1, \ldots, f_n, g_1, \ldots, g_m, C).$$

If each $(\mathrm{Pa}\,\mathcal{E})(A, B)$ is small we indeed obtain a category.

Consider the canonical $\mathcal{E}$-diagram $i_\mathcal{E} : \mathcal{E} \to \mathrm{Pa}\,\mathcal{E}$ which is tautological on objects as well as on paths of length 1. $\mathrm{Pa}\,\mathcal{E}$ has the following universal property: an $\mathcal{E}$-**diagram** d in any category $\mathcal{C}$ gives rise to a unique functor $d' : \mathrm{Pa}\,\mathcal{E} \to \mathcal{C}$ such that $d = d' \circ i_\mathcal{E}$.

3.2 Let Σ be a family of morphisms in category $\mathcal{C}$. If $J : \mathcal{C} \to \mathcal{D}$ is any functor let $\Sigma_* := J(\Sigma)$ be the class of all morphisms $J(f)$ where $f \in \Sigma$. Given $\mathcal{C}$ and Σ, consider the diagram scheme $\mathcal{E} = \mathcal{E}(\mathcal{C}, \Sigma)$ with $\mathrm{Ob}\,\mathcal{E} := \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{E} := \mathrm{Mor}\,\mathcal{C} \coprod \Sigma$, $\mathrm{s}_\mathcal{E} = \mathrm{s} \coprod \mathrm{s}|_\Sigma$, $\mathrm{T}_\mathcal{E} = \mathrm{T} \coprod \mathrm{T}|_\Sigma$. One has canonical inclusions $\mathrm{IN}_1 : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{E}$, $\mathrm{IN}_2 : \Sigma \hookrightarrow \mathrm{Mor}\,\mathcal{E}$. Let $\sim$ be the smallest equivalence relation on $\mathrm{Pa}\,\mathcal{E}$ such that

$$\begin{aligned}
(\mathrm{IN}_1 v) \circ (\mathrm{IN}_1 u) &\sim \mathrm{IN}_1(v \circ u) \text{ if } v \circ u \text{ is defined in } \mathcal{C}, \\
\mathrm{IN}_1(\mathrm{id}_A) &\sim (A, A), \quad A \in \mathcal{C}, \\
\left. \begin{aligned} (\mathrm{IN}_2 f) \circ (\mathrm{IN}_1 f) &\sim (\mathrm{s}(f), \mathrm{s}(f)) \\ (\mathrm{IN}_1 f) \circ (\mathrm{IN}_2 f) &\sim (\mathrm{T}(f), \mathrm{T}(f)) \end{aligned} \right\} & f \in \Sigma.
\end{aligned}$$

It is direct to show that operation $\circ$ induces a composition on classes of morphisms with respect to this particular equivalence relation. In this way

we obtain a quotient $\Sigma^{-1}\mathcal{C}$ of the category $\text{Pa}\,\mathcal{E}$ together with the canonical functor $Q_\Sigma^* : \mathcal{C} \to \text{Pa}\,\mathcal{E}$ which is tautological on objects and equals $i_\mathcal{E} \circ \text{IN}_1$ followed by the projection to the classes of equivalence on morphisms.

3.3 Proposition. *If $f \in \Sigma$ then the functor $Q_\Sigma^* : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$ sends f to an invertible map $Q_\Sigma^*(f)$. If $T : \mathcal{C} \to \mathcal{D}$ is any functor such that $T(s)$ is invertible for any $s \in \Sigma$ then there is a unique functor $H : \Sigma^{-1}\mathcal{A} \to \mathcal{B}$ such that $T = H \circ Q_\Sigma^*$.*

$\Sigma^{-1}\mathcal{C}$ is **category of fractions** of $\mathcal{C}$ at Σ. This construction has a defect, in that there is no general recipe to determine when two morphisms in $\text{Pa}\,\mathcal{E}$ represent the same morphism in $\Sigma^{-1}\mathcal{C}$. If Σ satisfies the Ore conditions, below, there is one.

3.4 Proposition. [43] *Let $T^* \dashv T_*$ be an adjoint pair of functors (this notation means that T^* is left adjoint to T_*), where $T^* : \mathcal{A} \to \mathcal{B}$. with adjunction counit $\epsilon : T^*T_* \Rightarrow 1_\mathcal{B}$. Let $\Sigma = \Sigma(T_*)$ be the class of all morphisms f in $\mathcal{A}$ such that $T^*(f)$ is invertible, and $Q_\Sigma^* : \mathcal{A} \to \Sigma^{-1}\mathcal{A}$ the natural functor. Then the following are equivalent:*

(i) T_* is fully faithful.

(ii) $\epsilon : T^*T_* \Rightarrow 1_\mathcal{B}$ is an isomorphism of functors.

(iii) The unique functor $H : \Sigma^{-1}\mathcal{A} \to \mathcal{B}$ such that $T^* = H \circ Q_\Sigma^*$ is an equivalence; in particular Q_Σ^* has a right adjoint $Q_{\Sigma*}$.

(iv) (If $\mathcal{A}$ is small) For each category $\mathcal{X}$, functor $\mathfrak{Cat}(-, \mathcal{X}) : \mathfrak{Cat}(\mathcal{B}, \mathcal{X}) \to \mathfrak{Cat}(\mathcal{A}, \mathcal{X})$ is fully faithful.

Throughout the paper, any functor T^* agreeing with a functor $Q_\Sigma^* : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$ as above up to category equivalences will be referred to as a **localization functor**. A functor T^* satisfying (i)-(iii) will be referred to as **a continuous localization functor**.

3.5 ([79]) *Suppose $Q^* : \mathcal{A} \to \mathcal{B}$ is a localization functor (cf. **3.4**), and $F : \mathcal{A} \to \mathcal{A}$ an endofunctor. If there is a functor $G : \mathcal{B} \to \mathcal{B}$ and a natural equivalence of functors $\alpha : Q^* \circ F \Rightarrow G \circ Q^*$ then there is a unique functor $F_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ such that $Q^* \circ F = F_\mathcal{B} \circ Q^*$. In that case, we say that F is **compatible with** Q^*.*

Proof. Suppose $f : M \to N$ is a morphism in $\mathcal{A}$. Suppose that $Q^*(f)$ is invertible. Then $GQ^*(f) : GQ^*(M) \to GQ^*(N)$ also has some inverse s. The naturality of α and α^{-1} implies

$$\alpha_M^{-1} \circ s \circ \alpha_N \circ Q^*F(f) = \alpha_M^{-1} \circ s \circ GQ^*(f) \circ \alpha_M = \text{id}_M,$$
$$Q^*F(f) \circ \alpha_M^{-1} \circ s \circ \alpha_N = \alpha_N^{-1} \circ GQ^*(f) \circ s \circ \alpha_N = \text{id}_N,$$

hence $\alpha_M^{-1} \circ s \circ \alpha_N : Q^*F(N) \to Q^*F(M)$ is the inverse of $Q^*F(f)$. The conclusion is that for any f with $Q^*(f)$ invertible, $Q^*F(f)$ is invertible as

well. In other words, (by the universal property of the localization), functor Q^*F factors through the quotient category $\mathcal{B}$, i.e. $\exists! F_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ with $Q^* \circ F = F_\mathcal{B} \circ Q^*$. Q.E.D.

4   Ore localization for monoids

4.1 A **semigroup** is a set R with a binary *associative* operation. A semigroup with unit element $1 \in R$ is called a **monoid**. By definition, maps of semigroups are set maps which respect the multiplication, and maps of monoids should preserve unit element as well. Monoids and maps of monoids form a category $\mathfrak{Mon}$, which has arbitrary products. The notion of a submonoid is the obvious one.

A subset S of a monoid R is **multiplicative** if $1 \in S$ and whenever $s_1, s_2 \in S$ then $s_1 s_2 \in S$. For a set $S_1 \subset R$ there is a smallest multiplicative subset $S \subset R$ containing S_1, namely the set of all products $s_1 \cdots s_n$ where $s_i \in S_1$, including the product of the empty set of elements which equals 1 by definition. We say that S is **multiplicatively generated** by S_1.

4.2 A multiplicative subset $S \subset R$ is a **left Ore set** if

- $(\forall s \in S \; \forall r \in R \; \exists s' \in S \; \exists r' \in R) \, (r's = s'r)$ (left Ore condition);

- $(\forall n_1, n_2 \in R \; \forall s \in S) \, (n_1 s = n_2 s) \Rightarrow (\exists s' \in S, \, s'n_1 = s'n_2)$

  (left reversibility).

4.3 In traditional ring-theoretic terminology, S is a left **Ore set** if the first condition holds and S is a **left denominator set** if both conditions hold. We often say "left Ore set" for a left denominator set, as is increasingly common among geometers, and the notion of satisfying just the left Ore condition may be said simply "satisfying left Ore condition". By saying (plural:) "left Ore condition*s*" we subsume both the left Ore condition and the left reversibility.

4.4 A monoid R can be viewed as a small category $\mathrm{Cat}(R)$ with a single object R. Left multiplication by an element $a \in R$ is a morphism in $\mathrm{Cat}R$ denoted by L_a. We compose the morphisms by composing the maps. Any small category having one single object is clearly equivalent to $\mathrm{Cat}(R)$ for a suitable R.

This suggests a generalization of the notion of a denominator set (as well as its applications below) by replacing $\mathrm{Cat}(R)$ by an arbitrary category. A **multiplicative system** in a category $\mathcal{A}$ is a class Σ of morphisms in $\mathcal{A}$ where all identity morphisms 1_A, where $A \in \mathrm{Ob}\,\mathcal{A}$, are in Σ, and for any

two composable morphism $s, t \in \Sigma$ (i.e., the target (range) of t matches the source (domain) of s), also $s \circ t \in S$.

A multiplicative system Σ satisfies the left Ore conditions if it satisfies the ordinary left Ore condition with all quantifiers conditioned on matching of the source and target maps appropriately.

More precisely, Σ satisfies the left Ore condition if

$$\forall (s : A \to B) \in \Sigma, \, \forall r : A' \to B, \, \exists (s' : D \to A') \in \Sigma, \, \exists r' : D \to A,$$

so that $r' \circ s = s' \circ r$. S satisfies left reversibility ('simplifiability') if

$$(\forall n_1, n_2 : A \to B, \, \forall (s : C \to A) \in \Sigma) \, (n_1 \circ s = n_2 \circ s)$$
$$\Rightarrow (\exists (s' : B \to D) \in \Sigma, \, s' \circ n_1 = s' \circ n_2).$$

We may picture the left simplifiability by the diagram

$$C \xrightarrow{\; s \;} A \underset{n_2}{\overset{n_1}{\rightrightarrows}} B \dashrightarrow^{s'} D$$

We say that Σ is a left denominator system, or equivalently, that the pair $(\mathcal{A}, \Sigma)$ forms a **left calculus of fractions** if the left Ore and left simplifiability condition hold. The book [104] has a good graphical treatment of that subject. See also [21, 35, 43, 103].

4.5 Lemma. *Let $f : R \to R'$ be a surjective map of monoids and $S \subset R$ left Ore. Then $f(S)$ is left Ore in R'.*

4.6 Let $\mathcal{D}$ be some category of monoids with additional structure, i.e. a category with a faithful functor $(-)_{\mathrm{mon}} : \mathcal{D} \to \mathfrak{Mon}$ preserving and reflecting finite equalizers. If R is an object in $\mathcal{D}$, a multiplicative set in R is by definition any multiplicative set $S \subset (R)_{\mathrm{mon}}$.

Definition. *Given a multiplicative set S in $R \in \mathcal{D}$ we introduce category $\mathcal{C}_{\mathcal{D}}(R, S)$ as follows. The objects of $\mathcal{C}_{\mathcal{D}}(R, S)$ are all pairs (j, Y) where $Y \in \mathrm{Ob}\,\mathcal{D}$ and $j : R \to Y$ is a morphism in $\mathcal{D}$ satisfying*

- $(\forall s \in S) \, (\exists u \in Y) \, (u j(s) = j(s) u = 1) \; in \; (Y)_{\mathrm{mon}};$

The morphisms of pairs $\sigma : (j, Y) \to (j', Y')$ are precisely those morphisms $\sigma : Y \to Y'$ in $\mathcal{D}$, for which $\sigma \circ j = j'$.

In plain words, we consider those morphisms which invert all $s \in S$.

Now we would like multiplying $j(s_1)^{-1} j(r_1) \cdot j(s_2)^{-1} j(r_2)$ to obtain again a 'left fraction' $j(s)^{-1} j(r)$. For this it is enough to be able to 'commute' the two middle terms in the sense $j(r_1) j(s_2)^{-1} = j(s')^{-1} j(r')$ as $j(s_1)^{-1} j(s')^{-1}) =$

$j(s's_1)^{-1}$ and $j(r')j(r_2) = j(r'r_2)$ and we are done. This reasoning is the origin of the left Ore condition. Here is a formal statement:

4.7 Proposition. *(i) For $(j, Y) \in \mathrm{Ob} \in \mathcal{C}_{\mathfrak{Mon}}(R, S)$, $j(S)$ is left Ore in $j(R)$ iff*

$$j(S)^{-1}j(R) = \{j(s)^{-1}j(r) \mid s \in S, r \in R\} \subset Y$$

is a submonoid of Y. In particular, if S is left Ore in R, $j(S)^{-1}j(R)$ is a submonoid of Y for each $(j, Y) \in \mathrm{Ob}(\mathcal{C}(R, S))$.
(ii) If the equivalent conditions in (i) hold, then

$$
\begin{array}{cc}
\begin{array}{c}
\forall (s, r) \in S \times R \\
j(s)^{-1}j(r) = j(s')^{-1}j(r')
\end{array}
\Leftrightarrow
\left\{
\begin{array}{l}
\exists \tilde{s} \in S, \exists \tilde{r} \in R, \\
j(\tilde{s})j(s') = j(\tilde{r})j(s) \\
j(\tilde{s})j(r') = j(\tilde{r})j(r).
\end{array}
\right.
\end{array}
\tag{1}
$$

Proof. (i) ($\Rightarrow$) Let $s_1, s_2 \in S$ $r_1, r_2 \in R$. By the left Ore condition $\exists s' \in S$ $\exists r' \in R$ with $j(s')j(r_1) = j(r')j(s_2)$. Hence the product $j(s_1)^{-1}j(r_1) \cdot j(s_2)^{-1}j(r_2) = j(s's_1)^{-1}j(r'r_2)$ belongs to Y.
($\Leftarrow$) If $j(S)^{-1}j(R)$ is a monoid then $j(r)j(s)^{-1} \in j(S)^{-1}j(R)$. In other words, $\exists s' \in S$ $\exists r' \in R$ such that $j(r)j(s)^{-1} = j(s')^{-1}j(r')$. Thus $j(s_1)j(r) = j(r_1)j(s)$.

(ii) By multiplying from the left by $j(s')$ one gets $j(s')j(s^{-1})j(r) = j(r')$. As S is left Ore, $\exists \tilde{s} \in S$ $\exists \tilde{r} \in R$ such that $\tilde{s}s' = \tilde{r}s$. This implies $j(\tilde{s})j(s') = j(\tilde{r})j(s)$ and, consequently, $j(s')j(s^{-1}) = j(\tilde{s})^{-1}j(\tilde{r})$; then $j(\tilde{s})^{-1}j(\tilde{r})j(r) = j(r')$ and, finally, $j(\tilde{r})j(r) = j(\tilde{s})j(r')$.

4.8 Proposition. Let S, R, Y, j be as in **4.7**, and let R, Y be each equipped with a binary operation, in both cases denoted by $+_0$, such that $\cdot$ is left distributive with respect to $+_0$. If $j(S)^{-1}j(R)$ is a submonoid of Y, then it is closed with respect to $+_0$ as well.

Proof. The following calculation is valid in Y:

$$
\begin{aligned}
j(s_1)^{-1}j(r_1) +_0 j(s_2)^{-1}j(r_2) &= j(s_1)^{-1}(j(r_1) +_0 j(s_1)j(s_2)^{-1}j(r_2)) \\
&= j(s_1)^{-1}(j(r_1) +_0 j(\tilde{s})^{-1}j(\tilde{r})j(r_2)) \\
&= j(\tilde{s}s_1)^{-1}(j(\tilde{s})j(r_1) +_0 j(\tilde{r})j(r_2)) \in j(S)^{-1}j(R),
\end{aligned}
$$

where $j(\tilde{s})j(s_1) = j(\tilde{r})j(s_2)$ for some $\tilde{s}, \tilde{r}$ by the left Ore condition which holds due **4.7**.

4.9 Remark. We do not require $j(a +_0 b) = j(a) +_0 j(b)$ here.

4.9a Exercise. Generalize this to a family $\mathcal{F}$ of n-ary left distributive operations in place of $+_0$, i.e., of operations of the form $F : X^{\times n} \to X$, for various $n \geq 0$, such that $L_a \circ F = F \circ L_a^{\times n}$.

4.10 From now on we limit to the case where the category $\mathcal{D}$ above corresponds to a variety $\mathfrak{D}$ of algebras $(A, \mathcal{L}_A)$ (in the sense of universal

algebra) of signature $\mathcal{L} = (\cdot, 1, \mathcal{F})$, where $\mathcal{F}_A$ is a family of left distributive operations on A on a $(A, \cdot_A, 1_A)$. The reader who does not care for this generality (suitable say for algebras with operators) can consider 3 basic cases: 1) $\mathcal{F} = \emptyset$ when $\mathcal{D} = \mathfrak{Mon}$; 2) $\mathcal{F} = \{+\}$ and algebras are unital rings; 3) $\mathcal{F} = \{+\}$ and algebras are associative unital **k**-algebras over a commutative ring **k**.

4.11 Denote by $\mathcal{C}_{l,\mathcal{D}}(R, S)$ the full subcategory of $\mathcal{C}_{\mathcal{D}}(R, S)$ consisting of those objects (j, Y) which satisfy

- $(\forall r, r' \in R)\,(j(r) = j(r') \Leftrightarrow \exists s \in S\,(sr = sr'))$.

- $j(S)^{-1}j(R)$ is a subring of Y

Similarly, $\mathcal{C}_{l,\mathcal{D}}^-(R, S)$ by definition consists of objects satisfying the first, but not necessarily the second property. Denote by $\mathcal{C}_{r,\mathcal{D}}(R, S)$ the full subcategory of $\mathcal{C}_{\mathcal{D}}(R, S)$ consisting of those objects (j, Y) which satisfy the symmetric conditions

- $(\forall r, r' \in R)\,(j(r) = j(r') \Leftrightarrow \exists s \in S\,(rs = r's))$.

- $j(R)j(S)^{-1}$ is a subring of Y

Finally, the objects in $\mathcal{C}_{r,\mathcal{D}}^-(R, S)$, by definition, satisfy the first, but not necessarily the second property. If there is a universal initial object in $\mathcal{C}_l(R, S)$ $(\mathcal{C}_r(R, S))$, we denote it by $(\iota, S^{-1}R)$ (resp.(ι, RS^{-1})) and we call the pair, or by abuse of language, also the ring $S^{-1}R$ $(RS^{-1}$ resp.), the left (right) **Ore localization** of R at set S, and map ι the localization map. An alternative name for $S^{-1}R$ (RS^{-1}) is the left (right) **ring of fractions** (of ring R at set S).

4.12 Proposition. *If $\forall(j, Y)$ in $\mathcal{C}_{l,\mathfrak{Mon}}^-(R, S)$ the subset $j(S)^{-1}j(R)$ is a submonoid (i.e.$\exists(j, Y) \in \mathcal{C}_{l,\mathfrak{Mon}}(R, S)$), then it is so $\forall(j, Y)$ in $\mathcal{C}_{l,\mathfrak{Mon}}^-(R, S)$, i.e. the categories $\mathcal{C}_{l,\mathfrak{Mon}}(R, S)$ and $\mathcal{C}_{l,\mathfrak{Mon}}^-(R, S)$ coincide. In that case, S is a left denominator set in R.*

Proof. Let $j(S)^{-1}j(R)$ be a subring for some (j, Y). Then $j(S)^{-1}j(R)$ is Ore in $j(R)$ by the previous proposition. Thus for every $s \in S$, $r \in R$ $\exists s' \in S$ $\exists r' \in R$ such that $j(r)j(s)^{-1} = j(s')^{-1}j(r')$ and therefore $j(s')j(r) = j(r')j(s)$ which means $j(s'r) = j(r's)$. That implies $\exists s^+ \in S$ with $s^+s'r = s^+r's)$. Therefore for any other (j', Y') in $\mathcal{C}_{l,\mathfrak{Mon}}(R, S)$ the subset $j'(S)^{-1}j'(R)$ is a subring. Moreover we have $s^+s' \in S$ and s^+r' satisfy $(s^+s')r = (s^+r')s$. Since they were constructed for an arbitrary s and r, S is left Ore in R.

Left reversibility: Let $r, r' \in R, s \in S$. Then $rs = r's \Rightarrow j(r)j(s) =$

$j(r')j(s)$, so by invertibility of $j(s)$ also $j(r) = j(r')$. But (j, Y) is object in $\mathcal{C}_{l,\mathfrak{Mon}}(R, S)$ so $j(r) = j(r') \Rightarrow \exists s' \in S, s'r = s'r'$.

**4.13 Lemma.** *(i) Let S be a left denominator set. Define the relation $\sim$ on $S \times R$ by*

$$(s, r) \sim (s', r') \;\Leftrightarrow\; (\exists \tilde{s} \in S \; \exists \tilde{r} \in R)\; (\tilde{s}s' = \tilde{r}s \;\; and \;\; \tilde{s}r' = \tilde{r}r).$$

Then $\sim$ is an equivalence relation.

(ii) Let Σ be a system of left fractions in a category $\mathcal{C}$. For any pair of objects X, Y in $\mathcal{C}$ let $(\Sigma \times \mathcal{C})(X, Y)$ be a class of all diagrams of the form $\left(X \xrightarrow{r} Z \xleftarrow{s} Y \right)$ in $\mathcal{C}$. Define a relation $\sim$ on $(\Sigma \times \mathcal{C})(X, Y)$ by

$$\left(X \xrightarrow{r} Z \xleftarrow{s} Y \right) \sim \left(X \xrightarrow{r'} Z' \xleftarrow{s'} Y \right)$$
$$\Leftrightarrow \exists \left(X \xleftarrow{\tilde{r}} B \xrightarrow{\tilde{s}} Y \right), \begin{cases} \tilde{s} \circ s' = \tilde{r} \circ s : B \to Z \\ \tilde{s} \circ r' = \tilde{r} \circ r : B \to Z' \end{cases}.$$

The latter condition can be depicted by saying that the diagram
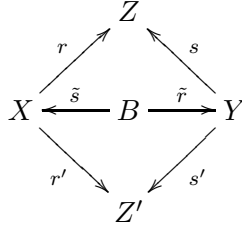
commutes. Conclusion: $\sim$ is an equivalence relation.

Here $\left(X \xleftarrow{\tilde{r}} B \xrightarrow{\tilde{s}} Y \right)$ is *not* a diagram in $(\Sigma^{\mathrm{op}} \times \mathcal{C})(X, Y)$.

Proof. Reflexivity is trivial.

Symmetry: By Ore $\exists r_1 \in R, s_1 \in S$ with $r_1 s = s_1 s'$. Also $\exists r_2 \in R, s_2 \in S$ with $r_1 s = s_1 s'$. Thus

$$r_2 \tilde{r} s' = r_2 \tilde{s} s = s_2 r_1 s = s_2 s_1 s'.$$

In other words $r_2 \tilde{r} - s_2 s_1 \in I_S$. Thus by the left reversibility, $\exists t \in S$ with $t(r_2 \tilde{r} - s_2 s_1) = 0$. Therefore $t(r_2 \tilde{r} - s_2 s_1)r' = 0$, hence $t s_2 s_1 r' = t r_2 \tilde{s} r$. Compare with $t s_2 s_1 s' = t r_2 \tilde{s} s$ to see that $(s', r') \sim (s, r)$.

Transitivity: Assume $(s, r) \sim (s', r')$ and $(s', r') \sim (s'', r'')$. This means

$$\exists \, \tilde{s}, \tilde{\tilde{s}} \in S \, \exists \, \tilde{r}, \tilde{\tilde{r}} \in R \quad \begin{cases} \tilde{s}s = \tilde{r}s' & \tilde{\tilde{s}}s' = \tilde{\tilde{r}}s'' \\ \tilde{s}r = \tilde{r}r' & \tilde{\tilde{s}}r' = \tilde{\tilde{r}}r'' \end{cases}$$

S is left Ore, hence $\exists s_* \in S \, \exists r_* \in R$ with $s_* \tilde{r} = r_* \tilde{\tilde{s}}$. Therefore

$$(s_* \tilde{s})s = s_* \tilde{r} s' = r_* \tilde{\tilde{s}} s' = (r_* \tilde{\tilde{s}}) s''$$
$$(s_* \tilde{s})r = s_* \tilde{r} r' = r_* \tilde{\tilde{s}} r' = (r_* \tilde{\tilde{s}}) r''$$

Hence $(s, r) \sim (s'', r'')$.

4.14 Simplifying observation. Consider a family of arrows $(s, r) \to ((ps)^{-1}, (pr))$ where $ps \in S$. Then $(s, r) \sim ((ps)^{-1}, (pr))$. If some property $\mathcal{P}$ of elements of $S \times R$ does not change along such arrows, then $\mathcal{P}$ is well-defined on classes $s^{-1}r := [s, r]/\sim \, \in S \times R/\sim$.

Proof. Clearly every $\sim$-arrow is a composition of one such arrow and an inverse of another such arrow.

4.15 Lemma. *If* $t \in S$ *and* $tr = tr'$ *(by reversibility even better if* $rt = r't$*) then* $(s, r_1 r r_2) \sim (s, r_1 r' r_2)$.

Proof. There are $t' \in S$, $r_1' \in R$ with $r_1' t = t' r_1$. Then $(s, r_1 r r_2) \sim (t's, t' r_1 r r_2) \sim (t's, r_1' t r r_2) \sim (t's, r_1' t r' r_2) \sim (t's, t' r_1 r' r_2) \sim (s, r_1 r' r_2)$.

4.16 Proposition. *For* (j, Y) *in* $\mathrm{Ob}\,\mathcal{C}_{l, \mathfrak{D}}(R, S)$ *the statement*

$$(\forall y \in Y \quad \exists s \in S \quad \exists r \in R) \, (y = j(s)^{-1} j(r)) \tag{2}$$

holds iff (j, Y) *is a universal initial object in* $\mathcal{C} = \mathcal{C}_{\mathfrak{D}}(R, S)$.

Proof. ($\Leftarrow$) Let $(j, Y) \in \mathcal{C}_l$ be universal in $\mathcal{C}$. Suppose $Y_0 := j(S)^{-1} j(R)$ is a proper subring of Y. We'll denote by j_0 the map from R to Y_0 agreeing with j elementwise. Then (j_0, Y_0) is an object in $\mathcal{C}_l$ and the inclusion $i : Y_0 \to Y$ is a morphism from (j_0, Y_0) into (j, Y). By universality of (j, Y) there is a morphism $i' : (j, Y) \to (j_0, Y_0)$. The composition of morphisms $i \circ i'$ is an automorphism of (j, Y) clearly different from the identity, contradicting the universality of (j, Y).

($\Rightarrow$) Let (j, Y) satisfy (2) and let (j', Y') be any object in $\mathcal{C}(R, S)$. We want to prove that there is unique map $i : Y \to Y'$ which satisfies $i(j(r)) = j'(r) \; \forall r \in R$. Note that $i(j(s)^{-1} j(s)) = i(j(s)^{-1}) j'(s)$ implies $i(j(s)^{-1}) = j'(s)^{-1}$. Thus $i(j(s)^{-1} j(r)) = j'(s)^{-1} j'(r)$ so that the value of i is forced for all elements in Y proving the uniqueness.

This formula sets i independently of choice of s and r. Indeed, if $j(s)^{-1} j(r) = j(s')^{-1} j(r')$ then $j(r) = j(s) j(s')^{-1} j(r')$. As $j(S)$ is left Ore in $j(R)$, we can find $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $j(\tilde{r}) j(s') = j(\tilde{s}) j(s)$ and therefore $j(s) j(s')^{-1} = j(\tilde{s})^{-1} j(\tilde{r})$. Thus $j(r) = j(\tilde{s})^{-1} j(\tilde{r}) j(r')$ or $j(\tilde{s}) j(r) = j(\tilde{r}) j(r')$ and, finally, $j(\tilde{s}r) = j(\tilde{r}r')$. Thus $\exists s^+ \in S$, $s^+ \tilde{s} r = s^+ \tilde{r} r'$. Starting here and reversing the chain of arguments, but with j' instead of j, we get $j'(s)^{-1} j'(r) = j'(s')^{-1} j'(r')$.

4.17 Theorem. *If S is a left denominator set in R, then the universal object (j, Y) in $\mathcal{C}_{l,\mathfrak{Mon}}(R, S)$ exists.*

Proof. We will construct a universal object $(j, Y) \equiv (\iota, S^{-1}R)$. <u>As a set,</u> $S^{-1}R := (S \times R)/\sim$. Let $[s, r]$, and, by abuse of notation, let $s^{-1}r$ also denote the $\sim$-equivalence class of a pair $(s, r) \in S \times R$. Notice that $1^{-1}r = 1^{-1}r'$ may hold even for some $r \neq r'$, namely when $\exists s \in S$ and r, r' with $sr = sr'$. The equivalence relation is forced by (1).

<u>Multiplication</u> $\boxed{s_1^{-1}r_1 \cdot s_2^{-1}r_2 := (\tilde{s}s_1)^{-1}(\tilde{r}r_2)}$ where $\tilde{r} \in R$, $\tilde{s} \in S$ satisfy $\tilde{r}s_2 = \tilde{s}r_1$ (thus $\tilde{s}^{-1}\tilde{r} = r_1 s_2^{-1}$), as in the diagram:
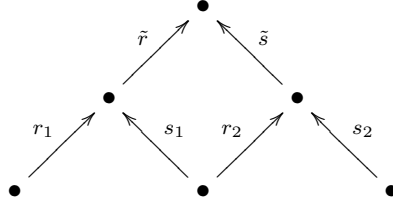
If we choose another pair of multipliers $\tilde{\tilde{r}} \in R, \tilde{\tilde{s}} \in S$ with $\tilde{\tilde{r}}s_2 = \tilde{\tilde{s}}r_1$ instead, then by the left Ore condition we can choose $r_* \in R, s_* \in S$ with $s_*\tilde{s} = r_*\tilde{\tilde{s}}$. Then

$$r_*\tilde{\tilde{r}}s_2 = r_*\tilde{\tilde{s}}r_1 = s_*\tilde{s}r_1 = s_*\tilde{r}s_2$$

and therefore $r_*\tilde{\tilde{r}} - s_*\tilde{r} \in I_S$.

In other words, $\exists s^+ \in S$ with $s^+ r_* \tilde{\tilde{r}} = s^+ s_* \tilde{r}$.

Therefore we have

$$s^+ s_* \tilde{r} r_2 = s^+ r_* \tilde{\tilde{r}} r_2$$
$$s^+ s_* \tilde{s} s_1 = s^+ r_* \tilde{\tilde{s}} s_1$$

which proves $(\tilde{s}s_1)^{-1}(\tilde{r}r_2) = (\tilde{\tilde{s}}s_1)^{-1}(\tilde{\tilde{r}}r_2)$. Thus multiplication is well defined as a map $\mu_1 : (S \times R) \times (S \times R) \to S^{-1}R$.

We have to show that μ_1 factors to $\mu : S^{-1}R \times S^{-1}R \to S^{-1}R$.

By **4.14**, it is sufficient to show that $a = \mu_1((s_1, r_1), (s_2, r_2))$ equals $b = \mu_1(((rs)_1, (rr_1)), ((ps)_2, (pr_2)))$ whenever $rs \in S$ and $ps \in S$.

$s_2' r_1 = r_1' s_2$ for some $s_2' \in S$ and $r_1' \in R$. Then $a = (s_2' s_1)^{-1}(r_1' r_2)$. As $ps_2 \in S$, $\exists p' \in R, s_* \in S$ with $p'(ps_2) = s_* r_1' s_2 = s_* s_2' r_1$. Furthermore, $s_\sharp r = p_\sharp s_* s_2'$ for some $s_\sharp \in S$ and $p_\sharp \in R$. Putting these together, we infer $s_\sharp(rr_1) = p_\sharp s_* s_2' r_1 = p_\sharp p' ps_2$ and therefore $(rr_1)(ps_2)^{-1} \to s_\sharp^{-1}(p_\sharp p')$, i.e., be definition, that $b = (s_\sharp rs_1)^{-1}(p_\sharp p' pr_2)$, hence by above, $b = (p_\sharp s_* s_2' s_1)^{-1}(p_\sharp p' pr_2)$. Now use lemma **4.15** and $(p'p)s_2 = (s_* r_1')s_2$ to conclude $b = (p_\sharp s_* s_2' s_1)^{-1}(p_\sharp s_* r_1' r_2) = (s_2' s_1)^{-1}(r_1' r_2) = a$.

Hence μ is well-defined. The unit element is clearly $1 = 1^{-1}1$. We need to show associativity of μ. The product $s_1^{-1}r_1 \cdot s_2^{-1}r_2 \cdot s_3^{-1}r_3$ does not depend on the bracketing, essentially because one can complete the following commutative diagram of elements in R (the composition is the multiplication in R: any pair of straight-line (composed) arrows with the same target is identified with a pair in $S \times R$):
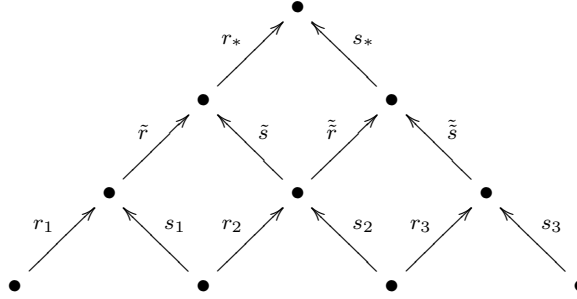
Finally, the construction gives the universal object because it clearly satisfies the equivalent condition in **4.16**.

5 Ore localization for rings

5.1 Exercise. The two left Ore conditions together immediately imply the **combined left Ore condition**:

If $n \in R$ is such that $ns = 0$ for some $s \in S$, then for every $r \in R$ there are $s' \in R$, $r' \in R$ such that $r'n = s'r$.

It is sometimes useful to quote this property in order to avoid introducing additional variables needed for deriving it.

5.2 Lemma. *Let $f : R \to R'$ be a ring morphism and $S \subset R$ Ore. Then $f(S)$ is an Ore set in R.*

5.3 Notation. In this section we are concerned only with the category of unital rings. Thus $\mathcal{C}(R,S) := \mathcal{C}_{Rings}(R,S)$.

5.4 Notation. For any $S \subset R$ let $I_S := \{n \in R \,|\, \exists s \in S, sn = 0\}$. I_S is clearly a right ideal. If S is a left Ore set, then $sn = 0$ and the left Ore condition imply that $\forall r \in R \; \exists s_0 \in S,\; r_0 \in R$ with $r_0 s = s_0 r$, hence $s_0 rn = r_0 sn = 0$. Thus I_S is then a 2-sided <u>ideal</u>.

5.5 Corollary. If $S^{-1}R$ exist then $\forall (j,Y) \in \mathcal{C}(R,S)$, $\ker j \subset I_S$. In particular, an Ore localization of a domain is a domain.

5.6 Theorem. *If S is a left denominator set in R then the universal object (j,Y) in $\mathcal{C}_l(R,S)$ exists.*

Proof. In **4.17**, we have constructed a monoid structure on $Y = S \times R/\sim$. We exhibit an additive structure on Y such that j is a ring map and (j, Y) is indeed universal.

Addition: Suppose we are given two fractions with representatives (s_1, r_1) and (s_2, r_2). By the left Ore condition, $\exists \tilde{s} \in S$, $\exists \tilde{r} \in R$, such that $\tilde{s} s_1 = \tilde{r} s_2$. The sum is then defined as

$$\boxed{s_1^{-1} r_1 + s_2^{-1} r_2 := (\tilde{s} s_1)^{-1} (\tilde{s} r_1 + \tilde{r} r_2)}$$

Suppose we have chosen $(\tilde{\tilde{s}}, \tilde{\tilde{r}}') \in S \times R$ with $\tilde{\tilde{s}} s_1 = \tilde{\tilde{r}} s_2$, instead of $(\tilde{s}, \tilde{r})$. Then by left Ore we find $s_* \in S$ and $r_* \in R$ such that $s_* \tilde{s} = r_* \tilde{\tilde{s}}$. Then

$$r_* \tilde{\tilde{r}} s_2 = r_* \tilde{\tilde{s}} s_1 = s_* \tilde{s} s_1 = s_* \tilde{r} s_2$$

hence $(s_* \tilde{r} - r_* \tilde{\tilde{s}}) \in I_S$, i.e. $\exists s^{\sharp} \in S$ with

$$s^{\sharp} s_* \tilde{r} = s^{\sharp} r_* \tilde{\tilde{s}}$$

Then

$$(s^{\sharp} s_*)(\tilde{s} r_1 + \tilde{r} r_2) = (s^{\sharp} r_*)(\tilde{\tilde{s}} r_1 + \tilde{\tilde{s}} r_2)$$
$$(s^{\sharp} s_*)(\tilde{s} s_1) = (s^{\sharp} r_*)(\tilde{\tilde{s}} s_2)$$

Conclusion: $(\tilde{s} s, \tilde{s} r_1 + \tilde{r} r_2) \sim (\tilde{\tilde{s}} s, \tilde{\tilde{s}} r_1 + \tilde{\tilde{r}} r_2)$, as required.

Now let's check that the sum does not depend on the choice of the representative of the first summand. Suppose we are given two representatives of the first fraction $s_1^{-1} r_1 = s_1'^{-1} r_1'$. Then for some $(s^*, r^*) \in S \times R$ we have

$$s_* s_1 = r_* s_1' \quad \text{and} \quad s_* r_1 = r_* r_1'$$

Second fraction in $s_2^{-1} r_2$. Choose

$$(\tilde{s}, \tilde{r}) \in S \times R \quad \text{with} \quad \tilde{s} s_1' = \tilde{r} s_2.$$

Now choose $(s_{\sharp}, r_{\sharp}) \in S \times R$ such that $s_{\sharp} r_* = r_{\sharp} \tilde{s}$. Then $(r_{\sharp} \tilde{r}) s_2 = r_{\sharp} \tilde{s} s_1' = s_{\sharp} r_* s_1' = (s_{\sharp} s_*) s_1$ and $(r_{\sharp} \tilde{s}) r_1' = s_{\sharp} r_* r_1' = (s_{\sharp} s_*) r_1$. Therefore

$$\begin{aligned}
s_1^{-1} r_1 + s_2^{-1} r_2 &= (s_{\sharp} s_* s_1)^{-1} (s_{\sharp} s_* r_1 + r_{\sharp} \tilde{r} r_2) \\
&= (r_{\sharp} \tilde{s} s_1')^{-1} (r_{\sharp} \tilde{s} r_1' + r_{\sharp} \tilde{r} r_2) \\
&= (\tilde{s} s_1')^{-1} (\tilde{s} r_1' + \tilde{r} r_2) \\
&= s_1'^{-1} r_1' + s_2^{-1} r_2
\end{aligned}$$

We should also check that the sum does not depend on the second summand. This proof would not be symmetric to this one as our definition of the sum

is not. We shall choose an indirect proof. Denote the set-theoretic quotient map by $p : S \times R \to S^{-1}R$. By now we have completed the proof that addition as a map from

$$\tilde{+} : S^{-1}R \times (S \times R) \to S^{-1}R$$

is well defined. Now we prove that the map

$$\tilde{+}(p \times \mathrm{id})\tau : (S \times R) \times (S \times R) \to S^{-1}R$$

where τ is the transposition of factors coincides with $\tilde{+}(p \times \mathrm{id})$. Thus we have a well-defined addition as a map defined on $S^{-1}R \times S^{-1}R$ which is then automatically commutative. It is sufficient to prove that for any two pairs (s_1, r_1) and (s_2, r_2) and any

$$\tilde{s}, \tilde{\tilde{s}} \in S, \tilde{r}, \tilde{\tilde{r}} \in R \;\; \text{with} \;\; \tilde{s}s_1 = \tilde{r}s_2, \quad \tilde{\tilde{r}}s_1 = \tilde{\tilde{s}}s_2,$$

the classes

$$(\tilde{s}s_1)^{-1}(\tilde{s}r_1 + \tilde{r}r_2)$$
$$(\tilde{\tilde{s}}s_2)^{-1}(\tilde{\tilde{r}}r_1 + \tilde{\tilde{s}}r_2)$$

coincide in $S^{-1}R$. For that purpose, choose $s_\sharp \in S$ and $r_\sharp \in R$ such that $s_\sharp\tilde{r} = r_\sharp\tilde{\tilde{s}}$. Then

$$r_\sharp\tilde{s}s_1 = s_\sharp\tilde{r}s_2 = r_\sharp\tilde{\tilde{s}}s_2.$$

Next $r_\sharp\tilde{\tilde{r}}s_1 = r_\sharp\tilde{\tilde{s}}s_2 = s_\sharp\tilde{r}s_2 = s_\sharp\tilde{s}s_1$, and therefore $(r_\sharp\tilde{\tilde{r}} - s_\sharp\tilde{s}) \in I_S$ (**5.4**). Thus $\exists s^+ \in S$ with

$$s^+r_\sharp\tilde{\tilde{r}} - s^+s_\sharp\tilde{s} = 0.$$

In particular, $s^+r_\sharp\tilde{\tilde{r}}r_1 = s^+s_\sharp\tilde{s}r_1 = 0$. Thus

$$
\begin{aligned}
(\tilde{s}s_1)^{-1}(\tilde{s}r_1 + \tilde{r}r_2) &= (s^+s_\sharp\tilde{s}s_1)^{-1}(s^+s_\sharp\tilde{s}r_1 + s^+s_\sharp\tilde{r}r_2) \\
&= (s^+r_\sharp\tilde{\tilde{s}}s_1)^{-1}(s^+r_\sharp\tilde{\tilde{r}}r_1 + s^+r_\sharp\tilde{\tilde{s}}r_2) \\
&= (\tilde{\tilde{s}}s_1)^{-1}(\tilde{\tilde{r}}r_1 + \tilde{\tilde{s}}r_2)
\end{aligned}
$$

The associativity of addition is left to the reader.

The distributivity law follows by **4.8**.

The element $1^{-1}0$ in $S^{-1}R$ is the zero and thus $S^{-1}R$ is equipped with a natural unital ring structure.

Define $\iota : R \to S^{-1}R$ by $\iota(r) = [1, r] = 1^{-1}r$. Check that ι is a unital ring homomorphism. Check that $\iota(S)$ consists of units and that $\iota(S)^{-1}\iota(R) = Y$. Pair $(\iota, S^{-1}R)$ is a universal object in $\mathcal{C}_l(R, S)$, as it clearly satisfies the equivalent condition in **4.16**.

5.7 *Right* Ore conditions, and right Ore localizations with respect to $S \subset R$, are by definition the left Ore conditions and localizations with respect to $S \subset R^{\mathrm{op}}$. The *right* ring of fractions is denoted $RS^{-1} := (S^{-1}R^{\mathrm{op}})^{\mathrm{op}}$. It consists of certain equivalence pairs $rs^{-1} := [(r, s)]$, where $(r, s) \in R \times S$.

6 Practical criteria for Ore sets

This section is to be read only by those who want to test in practice wheather they have an Ore set at hand.

6.1 Theorem. *(i) Let S and S' be multiplicative sets in ring R, where S is also left Ore in R. Assume*

1. for a map $j : R \to Y$ of unital rings, the image $j(S)$ consists of units in Y iff the image $j(S')$ consists of units in Y;

2. $sr = 0$ for some $s \in S$ iff $\exists s' \in S'$ with $s'r = 0$.

Then S' is left Ore as well and $S^{-1}R$ is canonically isomorphic to $S'^{-1}R$.
Proof. Under the assumptions the categories $\mathcal{C}_l(R,S)$ and $\mathcal{C}_l(R,S')$ are identical, so call them simply $\mathcal{C}$. The left Ore condition is equivalent to the existence of an initial object in $\mathcal{C}$; and the 2 localizations are just the 2 choices of an initial object, hence there is a unique isomorphism in $\mathcal{C}$ between them; its image under the forgetful functor $\mathcal{C} \to R - \mathrm{Mod}$ into the category of unital rings, is the canonical isomorphism as required.

6.2 The left Ore condition is often checked inductively on a filtration, or ordered set of generators. To disseminate this kind of reasoning we will temporarily use some *nonstandard* notation which generalizes the left Ore condition. One fixes a (only) multiplicative set $S \subset R$. For *any* subset $A \subset R$, and any $(s, r) \in S \times A \subset S \times R$, introduce predicate

$$\mathbf{lOre}(s, r \uparrow A) := \mathbf{lOre}_{S,R}(s, r \uparrow A) \equiv (\exists s' \in S, \exists r' \in A, s'r = r's),$$

where S, R in subscripts may be skipped if known from context. Moreover if $A = R$ then $\uparrow A$ may be skipped from the notation. For example, $\mathbf{lOre}(s, r) = \mathbf{lOre}_{S,R}(s, r \uparrow R)$.

For any subsets $A_0 \subset A$ and $S_0 \subset S$, abbreviate

$$\mathbf{lOre}(S_0, A_0 \uparrow A) \equiv (\forall s \in S_0, \forall r \in R_0, \mathbf{lOre}_{S,R}(s, r \uparrow A)),$$

with rules for skipping $\uparrow A$ as before. For example, $\mathbf{lOre}(S, R)$ is simply the left Ore condition for $S \subset R$.

Finally,
$$\mathbf{slOre}(S_0, A) \equiv \mathbf{lOre}_{S,R}(S_0, A \uparrow A).$$

For an additive subgroup $A \subset R$ consider also the relative versions, e.g.

$$\mathbf{rel} - \mathbf{lOre}_{S,R}(s, r; I) \equiv (\exists s' \in S, \exists r' \in R, s'r - r's \in A).$$

If $A = I$ is an ideal, then this predicate is suitable for study of some (non-Ore) generalizations (cf. [38] for such).

6.3 Extending Ore property *Let $A, B \subset R$ be additive subgroups of R, $A \subset B \subset R$, and $S \subset R$ multiplicative subset.*

(i) $(\mathbf{lOre}(S, A)$ and $\mathbf{rel} - \mathbf{lOre}(S, R; A)) \Rightarrow \mathbf{lOre}(S, R)$
(ii) $(\mathbf{lOre}(S, A \uparrow B)$ and $\mathbf{rel} - \mathbf{slOre}(S, B; A)) \Rightarrow \mathbf{lOre}(S, B)$
(iii) $(\mathbf{lOre}(S, A \uparrow B)$ and $\mathbf{rel} - \mathbf{slOre}(S, B; A)$
          and $SB \subset B) \Rightarrow \mathbf{slOre}(S, B)$

Proof. (i) is clearly the $B = R$ case of (ii). Let $b \in B$ and $s \in S$. Then $\mathbf{rel} - \mathbf{slOre}(S, B; A))$ means that $\exists s' \in S$, $\exists b' \in B$, $\exists a \in A$ such that $s'b - b's = a$. Now we compare s and a. There are $b_1 \in B$, $s_1 \in S$ such that $b_1 s = s_1 a$. Thus $s_1 s' b - s_1 b' s = s_1 a = b_1 s$, and finally, $(s_1 s')b = (s_1 b' + b_1)s$. S is multiplicative hence (ii), and if $SB \subset R$ then $s_1 b' + b_1 \in B$ hence (iii).

6.3a Remark.   The above condition is usually checked for generators only. Also we can iterate the above criterion if we have a finite or denumerable family of nested subrings, for which the induction is convenient. One may also need to nest subsets of S, with refined criteria, like $\mathbf{lOre}(S_1 \uparrow S_2, A_1 \uparrow A_2)$, where the $\uparrow S_2$ means that s' may be chosen in S_2.

6.4 Lemma. If S_1 *multiplicatively* generates S, and $A \subset R$ then

$$\mathbf{lOre}(S_1, R) \Leftrightarrow \mathbf{lOre}(S, R),$$

$$\mathbf{slOre}(S_1, A) \Leftrightarrow \mathbf{slOre}(S, A).$$

Proof.   The first statement is clearly a particular case of the second. Hence we prove the second statement; the nontrivial direction is $\Rightarrow$. By assumption, the set S can be written as a nested union $\cup_{n \geq 0} S_n$ where S_n consists of all those $s \in S$ which can be expressed as a product $\prod_{k=1}^{n'} s_k$ with $n' \leq n$ and $s_k \in S_k$; (hence S_1 is as the same as before). The assumption is $\mathbf{slOre}(S_1, A)$, hence by induction it is enough to prove that $\mathbf{slOre}(S_n, A) \Rightarrow \mathbf{slOre}(S_{n+1}, A)$ for all $n \geq 1$. Take $s = s_1 s_2 \cdots s_n$. Then $\mathbf{slOre}(S_n, A)$ means that for any $a \in A$ we have

$$\exists a' \in A \, \exists s' \in S \, (a' s_2 \cdots s_n = s'a),$$
$$\exists a'' \in A \, \exists s'' \in S \, (a'' s_1 = s''a')$$

and consequently

$$a'' s_1 s_2 \ldots s_n = s'' a' s_2 \ldots s_n = (s'' s')a,$$

with desired conclusion by the multiplicative closedness of S.

6.5 Lemma. If $A_0^+, A^+ \subset R$ be the additive closure of A_0, A respectively, then (obviously)

$$\mathbf{lOre}(S, A_0 \uparrow A) \Rightarrow \mathbf{lOre}(S, A_0^+ \uparrow A^+).$$

6.6 Lemma. If A generates R as a ring, then

$$\mathbf{lOre}(S, A) \Rightarrow \mathbf{lOre}(S, R).$$

Proof. By **6.5** it is enough to check this multiplicativity:

$$(\forall i, \mathbf{lOre}(S, c_i)) \Rightarrow \mathbf{lOre}(S, c_n \cdots c_1),$$

when $c_i \in A$. However, this general statement holds for any choice of c_i whatsoever. Namely, if we do not require $c_i \in A$, we see, by induction on n, that it is enough to prove this statement for $n = 2$. For $s \in S$ and $c_1, c_2 \in R$ we can find $r_1', r_2' \in R$ and $s', s'' \in S$, so that $r_1's = s'c_1$ and $r_2's' = s''c_2$. The result is

$$r_2'r_1's = r_2's'c_1 = s''c_2c_1,$$

hence the lemma is proved.

6.7 Theorem. *If $A \subset B$ and the subring $\bar{A} \subset R$ is also contained in B, then for any S_1 multiplicatively generating a multiplicatively closed set S we have*

$$\mathbf{slOre}(S_1, A) \Rightarrow \mathbf{lOre}(S, \bar{A} \uparrow B),$$

$$\mathbf{lOre}(S, A) \Rightarrow \mathbf{lOre}(S, \bar{A}).$$

Proof. We know $\mathbf{slOre}(S_1, A) \Rightarrow \mathbf{slOre}(S, A)$. Hence the first assertion follows from the second by $\bar{A} \subset B$. We proved the second statement in the case $\bar{A} = R$. If $S \subset \bar{A}$, the statement clearly does not say anything more than it would say after replacing R by its subring $\bar{A}$. The proof of the general case is exactly the same, as $s \in R$ is never used, and our calculations and quantifiers may be taken over a bigger overring.

6.8 Warning-theorem. *If A generates R as a ring and S_1 generates S multiplicatively, then it is NOT necessarily true that*

$$\mathbf{lOre}(S_1, A) \Rightarrow \mathbf{lOre}(S, R), \tag{3}$$

even if S_1 has only one multiplicative generator. We know from **6.4** that (3) holds if we replace **lOre** by the stronger condition **slOre**. Nevertheless, various intermediate conditions, standing between **lOre** and **slOre**, often utilizing filtrations and combined arguments, are widely used in practice. However it is also common to use (3) without proper justification.

6.8a Counterexample. (proving the warning statement above)

Let R be the unital ring generated by 4 generators z_1, z_2, z_3, D modulo the following relations:

$$Dz_1 = z_2 z_3 D,$$
$$D^2 z_2 = z_3 z_1 D, \quad\quad\quad (4)$$
$$D^3 z_3 = z_1 z_2 D.$$

Clearly the (4) are simply the identities needed to check $\mathbf{lOre}(S_1, A)$ where $S_1 = \{D\}$, and $A = \{z_1, z_2, z_3\}$. The powers of D on the left-hand side, which are $1, 2, 3$ may be replaced by $1, p, q$ respectively, where $p > 0$ and $q > 1$ are any integers, and the same proof applies, but the inequality $q > 1$ is indeed essential: if $p = q = 1$, this is not a counterexample at all.

We claim that for any nonnegative integer n, $D^n z_1 = PD^2$ does not have a solution for $P \in R$, hence the left Ore condition is *not* satisfied for the multiplicative set $S = \{1, D, D^2, \ldots\}$. The proof of the claim will be by contradiction, but we need first to study a convenient basis of ring R.

A basis of R consists of all ordered monomials where, the right-hand side of any of the equations (4) does not appear as a factor. This is obtained using SHIRSHOV-BERGMAN's diamond lemma ([15, 17]) with the reduction system K having 3 reductions corresponding to the relations (4) with production arrows from right to left. This reduction system has clearly no ambiguities whatsoever and all reductions send monomials into monomials in generators z_1, z_2, z_3, D. It is direct to see using this basis that D is not a zero divisor.

Suppose then, that S is left Ore. Then there exist n such that

$$D^n z_1 = PD^2, \quad \text{for some } P \in R. \quad\quad\quad (5)$$

We suppose that $n \geq 3$, and leave the remaining case to the reader. Equation (5) implies $D^{n-1} z_2 z_3 D = PD^2$. D is not a divisor, hence $D^{n-1} z_2 z_3 = PD$. Now write P as a sum of linearly independent K-reduced monomials P_i. Because D is not a zero divisor, monomials $P_i D$ are also linearly independent. Since the reductions in K send monomials to monomials, and $D^{n-1} z_2 z_3$ is a K-reduced monomial, we see that $D^{n-1} z_2 z_3$ can not be obtained as a sum of more than one of the linearly independent monomials $P_i D$, hence P has to be a monomial. The only way that PD in K-reduced form (which is $D^{n-1} z_2 z_3$) has z_3 as the most right-hand side factor is that $P = P' z_1 z_2$ for some P' in K-reduced form. Hence we obtain $D^{n-1} z_2 z_3 = P' z_1 z_2 D = P' D z_3$. Again using basis one can check directly that $Q z_3 = 0$ implies $Q = 0$, hence $D^{n-1} z_2 = P' D$. Now $D^{n-3} D^2 z_2 = D^{n-3} z_3 z_1 D = P' D$ implies $D^{n-3} z_3 z_1 = P'$. This substituted

back in the expression for P and the equation (5) gives

$$D^n z_1 = P' z_1 z_2 D = D^{n-3} z_3 z_1 z_1 z_2 D^2 = D^{n-1} z_2 D^2 z_3 D.$$

This is a contradiction as the two sides differ even though they are K-reduced.

6.9 Proposition. *Let S and T be left Ore sets in some ring R. Then the set of all elements of the form st where $s \in S$ and $t \in T$ satisfies the left Ore condition in R (but it is not necessarily multiplicatively closed).*

Proof. Suppose the contrary, i.e. there is $st \in ST$ and $r \in R$, such that we can not find $s' \in S, t' \in T$ and $r' \in B$, so that $r'st = s't'r$. Set T is Ore so there are $t' \in T$ and $r_1 \in R$ with $r_1 t = t'r$. Next we can find $s' \in S$ and $r_2 \in R$ so that $r_2 s = s' r_1$. Combining, we obtain

$$s't'r = s'r_1 t = r_2 st,$$

with contradiction.

6.10 Proposition. *Let S be a left Ore set in a k-algebra R. The set $1 \otimes S$ of all elements of $R \otimes_k R$ of the form $1 \otimes s$ where $s \in S$ satisfies the left Ore condition.*

Proof. $1 \otimes S$ is obviously multiplicatively closed.

If the Ore condition is not true, there is an element $y = \sum_{i=1}^{n} a_i \otimes b_i$ and an element $s \in S$ such that $(1 \otimes S)y \cap (R \otimes R)(1 \otimes s) = \emptyset$.

We use induction by n to find an element in the intersection. If $n = 1$ we simply use that S is left Ore to find $r' \in R$ and $s' \in S$ such that $r's = s'b_1$ and we have

$$(1 \otimes s')(a_1 \otimes b_1) = (a_1 \otimes r')(1 \otimes s),$$

which proves the basis of induction.

Suppose we found $s'_j \in S$ and $z = \sum_{i=1}^{j} a_i \otimes r'^j_i$ so that

$$(1 \otimes s'_j)(\sum_{i=1}^{j} a_i \otimes b_i) = (\sum_{i=1}^{j} a_i \otimes r'_i)(1 \otimes s).$$

Now we use again the property that S is left Ore to find $r'^{j+1}_{j+1} \in R$ and $\bar{s}_{j+1} \in S$ such that

$$r'^{j+1}_{j+1} s = \bar{s}_{j+1} s'_j b_{j+1}.$$

S is a multiplicatively closed set so $s'_{j+1} = \bar{s}_{j+1} s'_j$ is an element of S. Now

we have

$$
\begin{aligned}
(1 \otimes s'_{j+1})(\textstyle\sum_{i=1}^{j+1} a_i \otimes b_i) = \;\; & (1 \otimes \bar{s}_{j+1})(1 \otimes s'_j)(\textstyle\sum_{i=1}^{j} a_i \otimes b_i) + \\
& + (1 \otimes \bar{s}_{j+1})(1 \otimes s'_j)(a_{j+1} \otimes b_{j+1}) \\
= \;\; & (\textstyle\sum_{i=1}^{j} a_i \otimes \bar{s}_{j+1} r_i'^j)(1 \otimes s) + (a_{j+1} \otimes r_{j+1}'^{j+1})(1 \otimes s) \\
= \;\; & (\textstyle\sum_{i=1}^{j+1} a_i \otimes r_i'^{j+1})(1 \otimes s)
\end{aligned}
$$

where we denoted $r_i'^{j+1} = \bar{s}_{j+1} r_i'^j$ for $i < j+1$ and $r_{j+1}'^{j+1}$ has already been defined.

7 Ore localization for modules

The modern point of view on Ore localization is to express it as a *localization functor* on some category of modules. The localization map $\iota : R \to S^{-1}R$ will be replaced by a localization functor Q_S^* from R-Mod to $S^{-1}R$-Mod.

7.1 Let S be a left Ore set in a ring R, and M a left R-module. Notice that $S^{-1}R$ may be viewed as an $S^{-1}R$-R-bimodule. The **module of fractions** $S^{-1}M$ of M with respect to Ore set S is the left $S^{-1}R$-module

$$
Q_S^*(M) = S^{-1}M := S^{-1}R \otimes_R M. \tag{6}
$$

For each morphism $f : M \to N$, set $Q_S^*(f) := 1 \otimes f : S^{-1}R \otimes_R M \to S^{-1}R \otimes_R N$. This defines a localization functor $Q_S^* : R - \mathrm{mod} \to S^{-1}R - \mathrm{mod}$ whose right adjoint is the obvious forgetful functor $Q_{S*} : S^{-1}R - \mathrm{mod} \to R - \mathrm{mod}$. If $\iota = \iota_R : R \to S^{-1}R$ is the localization map, then define the map of R-modules $\iota_M : M \to S^{-1}M$ via $\iota_M = \iota_R \otimes_R \mathrm{id}$ i.e. $m \mapsto 1 \otimes m$, also called the localization map. Maps ι_M make together a natural transformation of functors, namely the adjunction $\iota : \mathrm{Id} \to Q_{S*}Q_S^*$.

7.1a Remark. If S is a right Ore set, and M a left R-module, then $M[S^{-1}] := R[S^{-1}] \otimes_R M$. If N is a right R-module then view RS^{-1} (or $S^{-1}R$) as a R-RS^{-1}- (resp. R-$S^{-1}R$)- bimodule and define $Q_S^*(N) := N \otimes_R R[S^{-1}]$ (resp. $N \otimes_R S^{-1}R$). We emphasize that the choice of right vs. left Ore sets is not correlated with the choice of right or left R-module categories, at least in the principle of the construction.

7.2 Universal property. For given R, M, S as above we define the category $\mathcal{M} = \mathcal{M}(R, M, S)$. The objects of $\mathcal{M}$ are pairs (N, h) where N is a left $S^{-1}R$-module and $h : M \to {}_R N$ a map of left R-modules. A morphism of pairs $\mu : (N, h) \to (N', h')$ is a map $\mu : N \to N'$ of $S^{-1}R$-modules such that $h' = \mu \circ h$.

Proposition. *The pair $(S^{-1}M, \iota_M)$ is an initial object in $\mathcal{M}$.*

Proof. For any pair $(N, h) \in \mathrm{Ob}(\mathcal{M})$ there is a well-defined morphism

$$\alpha : (S^{-1}M, \iota_M) \to (N, h) \ \text{ by } \ \alpha(s^{-1} \otimes_R m) = s^{-1}h(m). \qquad (7)$$

Let now α' be any morphism from $(S^{-1}M, \iota)$ to (N, h). By $h = \alpha' \circ i$ we conclude

$$\alpha'(1 \otimes m) = h(m) = \alpha(1 \otimes m), \ \forall m \in M.$$

The elements of the form $1 \otimes m$ generate $S^{-1}M$ as a module over $S^{-1}R$ and therefore $\alpha = \alpha'$.

7.3 Elements in the tensor product (6) are of the form $\sum_i s_i^{-1} \otimes m_i$ but such can be added up to a single term of that form, as the fractions can be always put to the common denominator. Namely, by the left Ore condition $\forall s, s' \in S \ \exists \tilde{s} \in S \ \exists \tilde{r} \in R, \ s's^{-1} = \tilde{s}^{-1}\tilde{r}$, and therefore

$$\begin{aligned} s^{-1} \otimes m + s'^{-1} \otimes m' &= s'^{-1}s's^{-1} \otimes m + s'^{-1}\tilde{s}^{-1}\tilde{s} \otimes m' \\ &= s'^{-1}\tilde{s}^{-1} \otimes (\tilde{r}m + \tilde{s}m'). \end{aligned} \qquad (8)$$

Thus every element in $S^{-1}M$ may be written in the form $s^{-1} \otimes_R m$, hence there is a surjection of sets $\nu : S \times M \to S^{-1}M$. The set $S \times M$ may be viewed as a set retract $S \times \{1\} \times M$ of $S \times R \times M$ via the retraction $(s, r, m) \mapsto (s, rm)$. Clearly ν extends to $\nu' : S \times R \times M \to S^{-1}M$. By the universality of the free abelian group $\mathbb{Z}(S \times R \times M)$ with basis $S \times R \times M$, $\exists! \tilde{\nu}' : \mathbb{Z}(S \times R \times M) \to S^{-1}M$ which is additive and extends ν'. It is clear by checking on the basis elements (s, r, m) and uniqueness that the composition of the canonical projections $\mathbb{Z}(S \times R \times M) \to \mathbb{Z}(S^{-1}R \times M) \to S^{-1}R \otimes_R M$ equals $\tilde{\nu}'$.

For $r \in R$ for which $rs \in S$, $s^{-1} \otimes_R m = (rs)^{-1}r \otimes_R m = (rs)^{-1} \otimes_R rm$ implying that $\ker \nu' \subset \mathbb{Z}(S \times M)$ contains all differences $(s, m) - (s', m')$ in $\mathbb{Z}(S \times M)$ of pairs in $S \times M$ which are equivalent via

$$\boxed{(s, m) \sim (s', m') \Leftrightarrow \exists r, r' \in R \quad rs = r's' \in S \text{ and } rm = r'm'.} \qquad (9)$$

Lemma. *(i) On $(S \times M)/\sim$ there is a unique binary operation $+$ such that*

$$(s, m) + (s, m') \sim (s, m + m'). \qquad (10)$$

(ii) $((S \times M)/\sim, +)$ is an abelian group. Hence by the universality of the free abelian group, the composition $S \times R \times M \to S \times M \to (S \times M)/\sim$ extends to a unique map $p : \mathbb{Z}(S \times R \times M) \to (S \times M)/\sim$ of abelian groups.

(iii) The map p factors to a map $p' : S^{-1}R \otimes_R M \to (S \times M)/\sim$.

(iv) p' is an inverse of ν, hence p' respects addition.

Proof. (i) *Uniqueness.* Suppose there are two such operations, $+_1, +_2$ and two classes (s_1, m_1) and (s_2, m_2) on which $+_1$ and $+_2$ disagree. By the left Ore condition choose $\tilde{s} \in S, \tilde{r} \in R$ with $\tilde{s}s_1 = \tilde{r}s_2$. Then $(s_1, m_1) +_i (s_2, m_2) \sim (\tilde{s}s_1, \tilde{s}m_1) +_i (\tilde{r}s_2, \tilde{r}m_2) \sim (\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2)$ which agree for $i = 1, 2$, giving a contradiction.

Existence. Take $(s_1, m_1) + (s_2, m_2) := (\tilde{r}_1 s_1, \tilde{r}_1 m_1 + \tilde{r}_2 m_2)$ with any choice of $\tilde{r}_1, \tilde{r}_2$ such that $\tilde{r}_1 s_1 = \tilde{r}_2 s_2 \in S$. We verify that the class of the result does not depend on the choices. If s_1, m_1 are replaced by $rs_1 \in S, rm_1 \in M$ we can by the combined Ore condition choose r_*, s_* with $s_* r = r_* \tilde{r}_1$, hence $S \ni s_* r s_1 = r_* \tilde{r}_1 s_1 = r_* \tilde{r}_2 s_2$. Hence the rule for the sum gives $(s_* r s_1, s_* r m_1 + r_* \tilde{r}_2 m_2) \sim (r_* \tilde{r}_1 s_1, r_* \tilde{r}_1 m_1 + r_* \tilde{r}_2 m_2) \sim (\tilde{r}_1 s_1, \tilde{r}_1 m_1 + \tilde{r}_2 m_2)$. By symmetry, we have the same independence for choice of (s_2, m_2). Finally, suppose that instead of choosing $\tilde{r}_1, \tilde{r}_2$ we choose $\tilde{b}_1, \tilde{b}_2$. As $\tilde{r}_1 s_1 \in S$. By the combined Ore condition we may choose $s_\sharp \in S$, $r_\sharp \in R$, such that $r_\sharp \tilde{r}_1 = s_\sharp \tilde{b}_1$ with $r_\sharp \tilde{r}_1 s_1 = s_\sharp (\tilde{b}_1 s_1) \in S$. Hence $(\tilde{r}_1 s_1, \tilde{r}_1 m_1 + \tilde{r}_2 m_2) \sim (r_\sharp \tilde{r}_1 s_1, r_\sharp \tilde{r}_1 m_1 + r_\sharp \tilde{r}_2 m_2) \sim (s_\sharp \tilde{b}_1 s_1, s_\sharp \tilde{b}_1 m_1 + r_\sharp \tilde{r}_2 m_2)$. Now $r_\sharp \tilde{r}_2 s_2 = r_\sharp \tilde{r}_1 s_1 = s_\sharp \tilde{b}_1 s_1 = s_\sharp \tilde{b}_2 s_2$, hence by left reversibility, there is s_2^* such that $s_2^* r_\sharp \tilde{r}_2 = s_2^* s_\sharp \tilde{b}_2$. Thus $(s_\sharp \tilde{b}_1 s_1, s_\sharp \tilde{b}_1 m_1 + r_\sharp \tilde{r}_2 m_2) \sim (s_2^* s_\sharp \tilde{b}_1 s_1, s_2^* s_\sharp \tilde{b}_1 m_1 + s_2^* s_\sharp \tilde{b}_2 m_2) \sim (\tilde{b}_1 s_1, \tilde{b}_1 m_1 + \tilde{b}_2 m_2)$, as required.

(ii) In the proof of existence in (i) we have seen that $+$ is commutative. Notice also that the class of $(s, 0)$ (independent on S) is the neutral element. For any pair, and hence for any triple of elements in $S \times M / \sim$, we can choose representatives such that all three are of the form (s, m) with the same s. For such triples the associativity directly follows by applying (10).

(iii) As p and the projection $S \times R \times M \to S \times M \to (S \times M) / \sim$ are additive it is sufficient to show that p sends the kernel of the projection to $0 \in (S \times M) / \sim$. The kernel of the projection is spanned by the elements of several obvious types, so we check for generators.

1. $(s, r, m) - (s', r', m)$ where $s^{-1} r = (s')^{-1} r'$. That means that for some $\tilde{s} \in S, \tilde{r} \in R$ we have $\tilde{s}s = \tilde{r}s'$ and $\tilde{s}r = \tilde{r}r'$. Compute $p(s, r, m) - p(s', r', m') = (s, rm) + (s', -r'm') = (\tilde{s}s, \tilde{s}rm) + (\tilde{r}s', -\tilde{r}r'm') = (\tilde{s}s, \tilde{s}rm - \tilde{r}r'm') = 0$.

2. Elements $(s, r + r', m) - (s, r, m) - (s, r + r', m)$, as well as $(s, rr', m) - (s, r, r'm)$ and $(s, r, m + m') - (s, r, m) - (s, r, m')$ go to 0 because, by (ii), in computing p one has to first act with the second component to the third.

7.4 Proposition. *$(S \times M) / \sim$ is additively canonically isomorphic to $S^{-1}M$. This isomorphism equips $(S \times M) / \sim$ with the canonical left $S^{-1}R$-module structure for which the following formulas can be taken as defining:*

$$t^{-1} r(s, m) = (s_* t, r_* m) \in (S \times M) / \sim$$

where $s_* \in S$ and $r_* \in R$ such that $s_* r = r_* s$.

Proof. By the lemma we have the first statement and hence we can view the class of (s, m) as $s^{-1} \otimes m$ and the formulas follow. They are defining because the formulas agree with the action transferred by an isomorphism, hence the existence, and by the left Ore condition for any $t^{-1} r$ and (s, m) there are s_*, r_* qualifying for the formulas, hence the uniqueness.

7.4a Corollary. *Let $S \subset R$ be left Ore and M a left R-module. Then $m = 0$ for some s iff $\exists s \in S$ and $sm = 0$.*

7.4b (long) **Exercise.** Let S, W be two left Ore sets in R, and $M = {}_R M$ a left R-module. The relation $\sim$ on $S \times W \times M$ given by

$$(s, w, m) \sim (s', w', m') \Leftrightarrow \begin{cases} \exists r, r', u, u', \tilde{r}, \tilde{r}' \in R, \exists \tilde{w}, \tilde{w}' \in W, \\ rs = r's' \in S, u\tilde{w} = u'\tilde{w}' \in W, \\ \tilde{w}r = \tilde{r}w, \tilde{w}'r' = \tilde{r}'w', z\tilde{r}m = z'\tilde{r}'m', \end{cases}$$

is an equivalence relation. Map $(S \times W \times M/ \sim) \to S^{-1} W^{-1} M$ given by $(s, w, m) \mapsto s^{-1} \otimes w^{-1} \otimes m$ is well-defined and bijective.

8 Monads, comonads and gluing

8.1 A monoidal category is a category $\mathcal{C}$ equipped with a 'tensor' product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$; a distinguished object $1_{\mathcal{C}}$, a family of associativity isomorphisms $c_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, natural in objects A, B, C in $\mathcal{C}$; left unit $l_A : 1_{\mathcal{C}} \otimes A \to A$, and right unit isomorphisms $r_A : A \otimes 1_{\mathcal{C}} \to A$, both indexed by and natural in objects A in $\mathcal{C}$; furthermore, require some standard coherence conditions (pentagon axiom for associativity coherence; left and right unit coherence conditions, cf. [21, 84]). A monoidal category $(\tilde{\mathcal{C}}, \otimes, 1_{\mathcal{C}}, c, r, l)$ is **strict** if c_{ABC}, l_A, r_A are actually all identity morphisms.

8.2 Monads and comonads. Given a diagram of categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, functors f_1, f_2, g_1, g_2 and natural transformations F, G as follows

$$\mathcal{A} \quad \begin{array}{c} \xrightarrow{f_1} \\ \Uparrow F \\ \xrightarrow{f_2} \end{array} \quad \mathcal{B} \quad \begin{array}{c} \xrightarrow{g_1} \\ \Uparrow G \\ \xrightarrow{g_2} \end{array} \quad \mathcal{C}, \tag{11}$$

one defines the natural transformation $G \star F : g_2 \circ f_2 \Rightarrow g_1 \circ f_1$ by

$$(G \star F)_A := G_{f_1(A)} \circ g_2(F_A) = g_1(F_A) \circ G_{f_2(A)} : g_2(f_2(A)) \to g_1(f_1(A)).$$

$(F, G) \mapsto F \star G$ is called the **Godement product** ('horizontal composition', cf. (11)). It is associative for triples for which $F \star (G \star H)$ is defined.

Given functors $f, g, h : \mathcal{A} \to \mathcal{B}$ and natural transformations $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, define their 'vertical' (or ordinary) composition $\beta \circ \alpha : f \Rightarrow h$ to be their composition taken objectwise: $(\beta \circ \alpha)_A := \beta_A \circ \alpha_A : f(A) \to h(A)$.

Note the **interchange law**: $(\alpha \circ \beta) \star (\gamma \circ \delta) = (\alpha \star \gamma) \circ (\beta \star \delta)$.

If T, T' are endofunctors in $\mathcal{A}$ and $\alpha : T \Rightarrow T'$, $\beta : T' \Rightarrow T$ natural transformations, one may also use the concatenation notation: $\alpha T : TT \Rightarrow T'T$ is given by $(\alpha T)_M := \alpha_{TM} : T(TM) \to T'(TM)$, i.e. $\alpha T \equiv \alpha \star 1_T$; similarly, $T\alpha$ equals $1_T \star \alpha$, or, $(T\alpha)_M = T(\alpha_M) : TTM \to TT'M$. This notation extends to the sequences with more functors but only one natural transformation – it is preferable to specify the product $\circ$ versus $\star$ among the words if each has some natural transformation mixed in. Here concatenation is higher binding than any of the composition signs. Notice that $1_T \star 1_S = 1_{T \circ S}$.

Given a strict monoidal category $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, 1_{\tilde{\mathcal{C}}})$ a **monoid** in $\mathcal{C}$ is a pair (X, μ) of an object X and a multiplication morphism $\mu : X \otimes X \to X$ which is associative and there is a 'unit' map $\eta : 1_{\tilde{\mathcal{C}}} \to X$ such that $\mu \circ (\eta \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \eta) \cong \mathrm{id}$ (here the identification $1_{\tilde{\mathcal{C}}} \otimes X \cong X$ is assumed). As this characterizes the unit map uniquely, one may or may not include η in the data, writing triples (X, μ, η) when convenient.

For a fixed small category $\mathcal{A}$, the category $\mathrm{End}_{\mathcal{A}}$ of endofunctors (as objects) and natural transformations (as morphisms) is a strict monoidal category: the product of endofunctors is the composition, the product of natural transformations is the Godement product, and the unit is $\mathrm{Id}_{\mathcal{A}}$.

A **monad** (T, μ, η) in an arbitrary category $\mathcal{A}$ is a monoid in $\mathrm{End}_{\mathcal{A}}$, and a **comonad** $(\bot, \delta, \eta)$ in $\mathcal{A}$ is a monad in $\mathcal{A}^{\mathrm{op}}$. The natural transformations $\delta : \bot \to \bot \circ \bot$ and $\epsilon : \bot \to \mathrm{Id}_{\mathcal{A}}$ are also called the coproduct and the counit of the comonad respectively.

An **action** of a monad $\mathbf{T} = (T, \mu, \eta)$ on an object M in $\mathcal{A}$ is a morphism $\nu : T(M) \to M$ such that the diagram

$$
\begin{array}{ccc}
TT(M) & \overset{\mu_M}{\Rightarrow} & T(M) \\
T(\nu) \downarrow & & \downarrow \nu \\
T(M) & \overset{\nu}{\to} & M
\end{array}
$$

commutes and $\nu \circ \eta_M = \mathrm{Id}_M$. We say that (M, μ) is a **module** (older terminology: algebra) over $\mathbf{T}$. A map $(M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in $\mathcal{A}$ intertwining the actions in the sense that $f \circ \nu_M = \nu_N \circ T(f) : T(M) \to N$. For a fixed $\mathbf{T}$, modules and their maps make a category $\mathcal{A}^{\mathbf{T}} \equiv \mathbf{T} - \mathrm{Mod}$, called the **Eilenberg-Moore** category of $\mathbf{T}$ ([37]). The natural forgetful functor $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$, $(M, \nu) \mapsto M$ is faithful, reflects isomorphisms and has a left adjoint $F : M \mapsto (M, \mu_M)$. The unit of adjunction $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow U^{\mathbf{T}} F = T$ coincides with the unit of $\mathbf{T}$, and the counit

$\epsilon : FU^{\mathbf{T}} \to \mathrm{Id}_{\mathcal{A}^{\mathbf{T}}}$ is given by $\epsilon_{(M,\nu)} = \nu$. The essential image of F, is a full and replete subcategory $\mathcal{A}_{\mathbf{T}} \subset \mathcal{A}^{\mathbf{T}}$ and it is called the **Kleisli category** of **T**, while its objects are called **free T-modules**.

Dually, for a comonad $\mathbf{G} = (G, \delta, \epsilon)$ in $\mathcal{A}$, a **G-comodule** is an object in category $\mathbf{G} - \mathrm{Comod} := ((\mathcal{A}^{\mathrm{op}})^{\mathbf{G}})^{\mathrm{op}} = (\mathbf{G} - \mathrm{Mod})^{\mathrm{op}}$; equivalently it is a pair (M, σ) where M is an object in $\mathcal{A}$ and $\sigma : M \to G(M)$ is a morphism in $\mathcal{A}$ such that

$$\begin{array}{ccc} M & \overset{\sigma}{\to} & G(M) \\ \sigma \downarrow & & \downarrow G(\sigma) \\ G(M) & \overset{\delta}{\to} & GG(M) \end{array}$$

and $\epsilon_M \circ \sigma = \mathrm{Id}_M$. A map $(M, \rho_M) \to (N, \nu_N)$ is a morphism $f : M \to N$ in $\mathcal{A}$ intertwining the coactions in the sense that $\sigma_N \circ f = T(f) \circ \sigma_M : M \to T(N)$. The forgetful functor $\mathbf{G} - \mathrm{Mod} \to \mathcal{A}^{\mathrm{op}}$ may be interpreted as a functor $U_{\mathbf{G}} : \mathbf{G} - \mathrm{Comod} \to \mathcal{A}$ which thus has a *right* adjoint $H : M \mapsto (G(M), \delta)$ whose essential image by definition consists of **cofree G-comodules**. The counit of the adjunction agrees with the counit of the comonad $\epsilon : G = U_{\mathbf{G}} H \Rightarrow \mathrm{Id}_{\mathcal{A}}$ and the unit $\eta : \mathrm{Id}_{\mathbf{G}-\mathrm{Comod}} \Rightarrow H U_{\mathbf{G}}$ is given by $\eta_{(M,\sigma)} = \sigma : (M, \sigma) \to (G(M), \delta)$.

8.3 An archetypal example of a monad is constructed from a pair of adjoint functors $Q^* \dashv Q_*$ where $Q_* : \mathcal{B} \to \mathcal{A}$. In other words there are natural transformations $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow Q_* Q^*$ called the unit and $\epsilon : Q^* Q_* \Rightarrow \mathrm{Id}_{\mathcal{B}}$, called the counit of the adjunction, such that the composites in the two diagrams

$$Q_* \overset{\eta Q_*}{\to} Q_* Q^* Q_* \overset{Q_*(\epsilon)}{\to} Q_*, \quad Q^* \overset{Q^*(\eta)}{\to} Q^* Q_* Q^* \overset{\epsilon Q^*}{\to} Q^*,$$

are the identity transformations. Then $\mathbf{T} := (Q_* Q^*, 1_{Q_*} \star \epsilon \star 1_{Q^*}, \eta)$ is a monad in $\mathcal{A}$. In other words, the multiplication is given by

$$\mu_M = Q_*(\epsilon_{Q^*(M)}) : Q_* Q^* Q_* Q^*(M) = TT(M) \to Q_* Q^*(M) = T(M).$$

The **comparison functor** $K^{\mathbf{T}} : \mathcal{B} \to \mathcal{A}^{\mathbf{T}}$ is defined by

$$M \mapsto (Q_*(M), Q_*(\epsilon_M)), \quad F \mapsto Q_*(f).$$

It is full and Q^* factorizes as $\mathcal{B} \overset{K^{\mathbf{T}}}{\to} \mathcal{A}^{\mathbf{T}} \overset{U^{\mathbf{T}}}{\to} \mathcal{A}$. More than one adjunction (varying $\mathcal{B}$) may generate the same monad in $\mathcal{A}$ in this vein.

Dually, $\mathbf{G} := (Q^* Q_*, Q^* \eta Q_*, \epsilon)$ is a comonad, i.e. a monad in $\mathcal{B}^{\mathrm{op}}$. The comparison functor $K^{\mathbf{G}} : \mathcal{A}^{\mathrm{op}} \to (\mathcal{B}^{\mathrm{op}})^{\mathbf{T}}$ is usually identified with a 'comparison functor' $K_{\mathbf{G}} : \mathcal{A} \to ((\mathcal{B}^{\mathrm{op}})^{\mathbf{T}})^{\mathrm{op}} \equiv \mathbf{G} - \mathrm{Comod}$ which is hence

given by $N \mapsto (Q^*(N), \eta_{Q^*(N)})$. $K_{\mathbf{G}}$ is full and Q_* factorizes as $\mathcal{A} \overset{K_{\mathbf{G}}}{\to}$ $\mathbf{G} - \mathrm{Comod} \overset{U_{\mathbf{G}}}{\to} \mathcal{B}$.

8.4 A map of monoids $f : (A, \mu, \eta) \to (A', \mu', \eta')$ in a monoidal category $(\mathcal{A}, \otimes, 1_{\mathcal{A}}, a, l, r)$ is a morphism $f : A \to A'$ in $\mathcal{A}$, commuting with multiplication: $\mu \circ (f \otimes f) = f \circ \mu$; and with the unit map: $\eta' \circ \mu = \eta \otimes \eta$, where on the left the application of one of the isomorphisms $l_{1_{\mathcal{A}}}, r_{1_{\mathcal{A}}} : 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} \to 1_{\mathcal{A}}$ is assumed. In particular, the morphism $\phi : (T, \mu, \eta) \to (T', \mu', \eta')$ of monads in $\mathcal{A}$ is a natural transformation $\phi : T \to T'$ such that $\mu \circ (\phi \star \phi) = \phi \circ \mu :$ $TT \Rightarrow T'$ and $\eta' \circ \mu = \eta \star \eta : TT \Rightarrow \mathrm{Id}_{\mathcal{A}}$. If M is an object in $\mathcal{A}$ and ν a T'-action on M, then $\nu' \circ \phi_M : TM \Rightarrow M$ is a T-action on M. More precisely, a natural transformation $\phi : T \Rightarrow T'$ and rules $\mathcal{A}^{\phi}(M, \nu) = (M, \nu' \circ \phi_M)$ and $\mathcal{A}^{\phi}(f) = f$ define a functor

$$\mathcal{A}^{\phi} : \mathcal{A}^{\mathbf{T}'} \to \mathcal{A}^{\mathbf{T}}$$

iff ϕ is a morphism of monads and every functor $\mathcal{A}^{\mathbf{T}'} \to \mathcal{A}^{\mathbf{T}}$ inducing the identity on $\mathcal{A}$ is of that form.

8.5 Let $\boldsymbol{\Delta}$ be the 'simplicial' category: its objects are nonnegative integers viewed as finite ordered sets $\mathbf{n} := \{0 < 1 < \ldots < n\}$ and its morphisms are nondecreasing monotone functions. Given a category $\mathcal{A}$, denote by $\mathrm{Sim}\mathcal{A}$ the category of **simplicial objects** in $\mathcal{A}$, i.e. functors $F : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{A}$. Represent F in $\mathrm{Sim}\mathcal{A}$ as a sequence $F_n := F(\mathbf{n})$ of objects, together with the face maps $\partial_i^n : F_n \to F_{n-1}$ and the degeneracy maps $\sigma_i^n : F_n \to F_{n+1}$ for $i \in \mathbf{n}$ satisfying the familiar simplicial identities ([144, 84]). Notation $F_{\bullet}$ for this data is standard.

Given a comonad $\mathbf{G}$ in $\mathcal{A}$ one defines the sequence $\mathbf{G}_{\bullet}$ of endofunctors $\mathbb{Z}_{\geq 0} \ni n \mapsto \mathbf{G}_n := G^{n+1} := G \circ G \circ \ldots \circ G$, together with natural transformations $\partial_i^n : G^i \epsilon G^{n-i} : G^{n+1} \to G^n$ and $\sigma_i^n : G^i \delta G^{n-i} : G^{n+1} \to G^{n+2}$, satisfying the simplicial identities. Use $\epsilon G \circ \delta = G\epsilon \circ \delta = \mathrm{Id}_{\mathcal{A}}$ in the proof. Hence any comonad $\mathbf{G}$ canonically induces a simplicial endofunctor, i.e. a functor $\mathbf{G}_{\bullet} : \Delta^{\mathrm{op}} \to \mathrm{End}\mathcal{A}$, or equivalently, a functor $\mathbf{G}_{\bullet} : \mathcal{A} \to \mathrm{Sim}\mathcal{A}$. The counit ϵ of the comonad $\mathbf{G}$ satisfies $\epsilon \circ \partial_0^1 = \epsilon \circ \partial_1^1$, hence $\epsilon : \mathbf{G}_{\bullet} \to \mathrm{Id}_{\mathcal{A}}$ is in fact an augmented simplicial endofunctor.

This fact is widely used in homological algebra ([8, 84, 144]), and now also in the cohomological study of noncommutative spaces ([109]).

8.6 Barr-Beck lemma. ([9, 85]) Let $Q^* \dashv Q_*$ be an adjoint pair $\mathbf{T}$ its associated monad, and $\mathbf{G}$ its associated comonad (as in **8.3**). Recall the notions of preserving and reflecting (co)limits from **1.6**.

If Q_* preserves and reflects coequalizers of all parallel pairs in $\mathcal{A}$ (for which coequalizers exists) and if any parallel pair mapped by Q_* into a pair

having a coequalizer in $\mathcal{B}$ has a coequalizer in $\mathcal{A}$ then the comparison functor $K : \mathcal{B} \to \mathcal{A}^{\mathbf{T}}$ is an equivalence of categories.

If Q^* preserves and reflects equalizers of all parallel pairs in $\mathcal{B}$ (for which equalizers exists) and if any parallel pair mapped by Q^* into a pair having an equalizer in $\mathcal{A}$ has an equalizer in $\mathcal{B}$ then the comparison functor $K' : \mathcal{A} \to \mathbf{G} - \mathrm{Comod}$ is an equivalence of categories.

Left (right) exact functors by definition preserve finite limits (colimits) and faithful functors clearly reflect both. In particular, this holds for (co)equalizers of parallel pairs. In abelian categories such (co)equalizers always exist. Hence

Corollary. *Consider an adjoint pair $Q^* \dashv Q_*$ of additive functors between abelian categories. If Q_* is faithful and exact, then the comparison functor for the associated comonad is an equivalence. If Q^* is faithful and exact, then the comparison functor for the associated monad is an equivalence.*

Given a functor $U : \tilde{\mathcal{M}} \to \mathcal{M}$ one may ask when there is a monad $\mathbf{T}$ in $\mathcal{M}$ and an equivalence $H : \tilde{\mathcal{M}} \to \mathcal{M}^{\mathbf{T}}$ such that $U^T H = U$. The conditions are given by the *Beck monadicity (=tripleability) theorem(s)* ([9, 11, 84, 85]). If we already know that U has left adjoint, this may be rephrased by asking if the comparison functor for the *associated* monad is an equivalence. Barr-Beck lemma gives only sufficient conditions for this case, it is easier to use, and widely applicable.

8.7 ([109]) **A comonad associated to a family of continuous functors.** Let $\{Q_\lambda^* : \mathcal{A} \to \mathcal{B}_\lambda\}_{\lambda \in \Lambda}$ be a small family of continuous (= having a right adjoint) functors. The categories $\mathcal{B}_\lambda$ are *not* necessarily constructed from $\mathcal{A}$ by a localization.

One may consider the category $\mathcal{B}_\Lambda := \prod_{\lambda \in \Lambda} \mathcal{B}_\lambda$ whose objects are families $\prod_{\lambda \in \Lambda} M^\lambda$ of objects M^λ in $\mathcal{B}_\lambda$ and morphisms are families $\prod_{\lambda \in \Lambda} f_\lambda : \prod_{\lambda \in \Lambda} M^\lambda \to \prod_{\lambda \in \Lambda} N^\lambda$ where $f_\lambda : M^\lambda \to N^\lambda$ is a morphism in $\mathcal{B}_\lambda$, with componentwise composition. This makes sense as the family of objects is literally a function from Λ to the disjoint union $\coprod_\lambda \mathrm{Ob}\,\mathcal{B}_\lambda$ which is in the same Grothendieck universe.

The family of adjoint pairs $Q_\lambda^* \dashv Q_{\lambda *}$ defines an inverse image functor $\mathbf{Q}^* = \prod Q_\lambda^* : \mathcal{A} \to \mathcal{A}_\Lambda$ by $\mathbf{Q}^*(M) := \prod_{\lambda \in \Lambda} Q_\lambda^*(M)$ on objects and $\mathbf{Q}^*(f) := \prod_\lambda Q_\lambda^*(f)$ on morphisms. However, a direct image functor may not exist. We may naturally try $\mathbf{Q}_* : \prod_\lambda' M^\lambda \mapsto \prod_\lambda' Q_*(M^\lambda)$ where $\prod'$ is now the symbol for the *Cartesian* product in $\mathcal{A}$ which may not always exist. For *finite* families, with $\mathcal{A}$ abelian, these trivially exist. Let $\mathcal{A}^\Lambda = \prod_{\lambda \in \Lambda} \mathcal{A}$ be the power category. Assume a fixed choice of the Cartesian product for all

Λ-tuples in $\mathcal{A}$. Then $\{M^\lambda\}_\lambda \mapsto \prod'_{\lambda \in \Lambda} M^\lambda$ extends to a functor $\mathcal{A}^\Lambda \to \mathcal{A}$, and the universality of products implies that the projections $p'_{\nu \{M^\lambda\}_\lambda} : \prod_\lambda M^\lambda \to M^\nu$ form a natural transformation of functors $p'_\nu : \prod'_\lambda \mathrm{Id}_\mathcal{A} \Rightarrow \mathrm{p}_\nu$ where $p_\nu : \mathcal{A}^\Lambda \to \mathcal{A}$ is the ν-th formal projection $\prod_\lambda M^\lambda \to M^\nu$. The unique liftings $\boldsymbol{\eta}_M : M \to \mathbf{Q}_*\mathbf{Q}^*(M)$ of morphisms $\eta_{\nu M} : M \to Q_{\nu *}Q_\nu^*(M)$ in the sense that $(\forall \nu)\, \eta_{\nu M} = p'_{\nu M} \circ \eta_M$ hence form a natural transformation $\boldsymbol{\eta} : \mathrm{Id}_\mathcal{A} \Rightarrow \mathbf{Q}_*\mathbf{Q}^*$.

Define $\boldsymbol{\epsilon} \equiv \prod_\lambda \epsilon_\lambda : \prod_\lambda Q_\lambda^* Q_{\lambda *} \Rightarrow \prod_\lambda \mathrm{Id}_{\mathcal{B}_\lambda} = \mathrm{Id}_{\mathcal{B}_\Lambda}$ componentwise. This way we obtain an adjunction $\mathbf{Q}^* \dashv \mathbf{Q}_*$. If $Q_{\lambda *}$ is faithful and exact for every λ then $\mathbf{Q}_*$ is as well.

Consider the comonad $\mathbf{G}$ in $\mathcal{B}_\Lambda$ associated to $\mathbf{Q}^* \dashv \mathbf{Q}_*$. We are interested in situation when the comparison functor $K_\mathbf{G}$ is an equivalence of categories. That type of a situation arises in practice in two different ways:

- 1) All categories $\mathcal{A}, \mathcal{B}_\lambda$ and flat localization functors $Q_\lambda^*, Q_{\lambda *}$ are given at start and the construction is such that we know the faithfulness of $\mathbf{Q}_*$.

- 2) Only categories $\mathcal{B}_\lambda$ are given (not $\mathcal{A}$) but equipped with **gluing morphisms** i.e. the family Φ of flat functors (not necessarily localizations) $\phi_{\lambda,\lambda'}^* : \mathcal{B}_\lambda \to \mathcal{B}_{\lambda'}$ for each pair λ, λ', where Φ satisfies some cocycle condition.

8.7a In 1), to ensure the faithfulness of $\mathbf{Q}_*$ we require that the family $\{Q_\lambda^*\}_{\lambda \in \Lambda}$ is a flat **cover** of $\mathcal{A}$. That means that this is a small *flat* family of functors with domain $\mathcal{A}$ which is **conservative**, i.e. a morphism $f \in \mathcal{A}$ is invertible iff $Q_\lambda^*(f)$ is invertible for each $\lambda \in \Lambda$. A flat map whose direct image functor is conservative is called **almost affine**. In particular, this is true for adjoint triples $f^* \dashv f_* \dashv f^!$ coming from a map $f : R \to S$ of rings. Adjoint triples where the direct image functor is conservative are called **affine morphisms**.

8.7b In 2), we *a posteriori* construct $\mathcal{A}$ to be $\mathcal{B}_\Lambda$ as before but equipped with functors $Q_\lambda^* \dashv Q_{\lambda *}$ constructed from ϕ-functors. The cocycle condition for gluing morphism is equivalent to the associativity of the associated comonad ([21]). The remaining requirements are made to ensure that the comparison functor is an equivalence and the other original data may be reconstructed as well. The Eilenberg-Moore category of the associated monad may be constructed directly from gluing morphisms, and it appears to be just a reformulation of the descent category. In a generalization, the category $\mathcal{B}_{\lambda\lambda'}$ which is the essential image of $\phi_{\lambda\lambda'}$ in $\mathcal{B}_{\lambda'}$ may be

replaced by any 'external' category $\mathcal{B}_{\lambda\lambda'}$, but then, instead of $\phi^*_{\lambda\lambda'}$ one requires not only flat functors $\phi^*_{\lambda\lambda'} =: \phi^{\lambda*}_{\lambda\lambda'} : \mathcal{B}_\lambda \to \mathcal{B}_{\lambda\lambda'}$, but also flat functors $\tilde{\phi}^*_{\lambda\lambda'} := \phi^{\lambda'*}_{\lambda\lambda'} : \mathcal{B}_{\lambda'} \to \mathcal{B}_{\lambda\lambda'}$. This generalization is essentially more general only if we allow the direct image functors (of the second type, i.e. $\phi^{\lambda'}_{\lambda\lambda'*}$), to be not necessarily fully faithful (hence $\mathcal{B}_{\lambda\lambda'}$ may not be viewed as a full subcategory of $\mathcal{B}_{\lambda'}$). Another generalization of this descent situation, which can be phrased as having a pseudofunctor from a finite poset Λ (viewed as a 2-category with only identity 2-cells) to the 2-category of categories has been studied by V. LUNTS ([78]). This analogue of the descent category is called a *configuration category*.

8.7c The usual formalism of descent is via fibred categories, cf. [142]. For the correspondence between the two formalisms see, e.g. [110].

8.8 Globalization lemma. (version for Gabriel filters: in [112] p. 103) *Suppose* $\{Q^*_\lambda : R - \mathrm{Mod} \to \mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ *is a finite cover of* $R - \mathrm{Mod}$ *by flat localization functors (e.g. a conservative family of Ore localizations* $\{S^{-1}_\lambda R\}_{\lambda\in\Lambda}$*). Denote* $Q_\lambda := Q_{\lambda*}Q^*_\lambda$ *where* $Q_{\lambda*}$*, is the right adjoint to* Q^*_λ*. Then for every left R-module M the sequence*

$$0 \to M \xrightarrow{\iota_{\Lambda,M}} \prod_{\lambda\in\Lambda} Q_\lambda M \xrightarrow{\iota_{\Lambda\Lambda,M}} \prod_{(\mu,\nu)\in\Lambda\times\Lambda} Q_\mu Q_\nu M$$

is exact, where $\iota_{\Lambda M} : m \mapsto \prod \iota_{\lambda,M}(m)$*, and*

$$\iota_{\Lambda\Lambda M} := \prod_\lambda m_\lambda \mapsto \prod_{(\mu,\nu)} (\iota^\mu_{\mu,\nu,M}(m_\mu) - \iota^\nu_{\mu,\nu,M}(m_\nu)).$$

Here the order matters: pairs with $\mu = \nu$ may be (trivially) skipped, but, unlike in the commutative case, **we can not confine to the pairs of indices with** $\mu < \nu$ **only**. *Nota bene!*

Proof. A direct corollary of Barr-Beck lemma. For proofs in terms of Gabriel filters and torsion see [113], pp. 23–25, and [62, 137, 138].

8.9 A monad $\mathbf{T} = (T, \mu, \eta)$ in $\mathcal{A}$ is **idempotent** if the multiplication $\mu : TT \Rightarrow T$ is an equivalence of endofunctors. As μ_M is the left inverse of η_{TM}, and of $T(\eta_M)$, then μ_M is invertible iff any of them is, hence both, and then $\eta_{TM} = T(\eta_M) = \mu^{-1}_M$.

If $\nu : TM \to M$ is a $\mathbf{T}$-action, then by naturality $\eta_M \circ \nu = T(\nu) \circ \eta_{TM} = T(\nu) \circ T(\eta_M) = T(\nu \circ \eta_M) = \mathrm{Id}_{TM}$, hence η_M is 2-sided inverse of ν in $\mathcal{A}$, hence *every* $\mathbf{T}$-*action is an isomorphism*. Conversely, If every $\mathbf{T}$-action is an isomorphism, μ_M is in particular, and $\mathbf{T}$ is idempotent. Moreover, if every action $\nu : TM \to M$ is an isomorphism, then its right inverse η_M must be

the 2-sided inverse, hence there may be *at most* one action on a given object M in $\mathcal{A}$. By naturality of η, its inverse ν which is well-defined on the full subcategory of $\mathcal{A}$ generated by objects in the image of $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$, is also natural, i.e. it intertwines the actions, hence it is in fact a morphism in $\mathcal{A}^{\mathbf{T}}$, hence the forgetful functor $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ is not only faithful but also full. Its image is strictly full (full and closed under isomorphisms) as the existence of $\mathbf{T}$-actions on M depends only on the isomorphism class of M in $\mathcal{A}$. To summarize, $\mathcal{A}^{\mathbf{T}}$ includes via $U^{\mathbf{T}}$ as a strictly full subcategory into $\mathcal{A}$ and the inclusion has a left adjoint $F^{\mathbf{T}}$.

In general, a **(co)reflective subcategory** $\mathcal{B} \hookrightarrow \mathcal{A}$ is a strictly full subcategory, such that the inclusion $U : \mathcal{B} \hookrightarrow \mathcal{A}$ has a (right) left adjoint, say F. As $F^{\mathbf{T}} \dashv U^{\mathbf{T}}$, we have just proved that $\mathcal{A}^{\mathbf{T}}$ is canonically isomorphic to a reflective subcategory of $\mathcal{A}$ via inclusion $U^{\mathbf{T}}$ if $\mathbf{T}$ is idempotent. On the other hand, it may be shown that for any reflective subcategory $U : \mathcal{B} \hookrightarrow \mathcal{A}$ the corresponding monad $(UF, U(\epsilon_F), \eta)$ is idempotent and the comparison functor $\mathcal{K} : \mathcal{B} \cong \mathcal{A}^{\mathbf{T}}$ is an isomorphism. Similarly, coreflective subcategories are in a natural correspondence with idempotent comonads.

9 Distributive laws and compatibility

9.1 A **distributive law from a monad** $\mathbf{T} = (T, \mu^T, \eta^T)$ **to an endofunctor** P is a natural transformation $l : TP \Rightarrow PT$ such that

$$l \circ (\eta^T)_P = P(\eta^T), \qquad l \circ (\mu^T)_P = P(\mu^T) \circ l_T \circ T(l). \tag{12}$$

Then P **lifts** to a unique endofunctor $P^{\mathbf{T}}$ in $\mathcal{A}^{\mathbf{T}}$, in the sense that $U^{\mathbf{T}} P^{\mathbf{T}} = PU^{\mathbf{T}}$. Indeed, the endofunctor $P^{\mathbf{T}}$ is given by $(M, \nu) \mapsto (PM, P(\nu) \circ l_M)$.

9.1a A **distributive law from a monad** $\mathbf{T} = (T, \mu^T, \eta^T)$ **to a monad** $\mathbf{P} = (P, \mu^P, \eta^P)$ **in** $\mathcal{A}$ ([10]) (or "of $\mathbf{T}$ over $\mathbf{P}$") is a distributive law from $\mathbf{T}$ to the endofunctor P, compatible with μ^P, η^P in the sense that

$$l \circ T(\eta^P) = (\eta^P)_T, \qquad l \circ T(\mu^P) = (\mu^P)_T \circ P(l) \circ l_P.$$

For clarity, we show the commutative diagram for one of the relations.

$$
\begin{array}{ccccc}
TPP & \xrightarrow{l_P} & PTP & \xrightarrow{P(l)} & PPT \\
{\scriptstyle T(\mu^P)}\downarrow & & & & \downarrow{\scriptstyle P(\mu^T)} \\
TP & & \xrightarrow{\quad l \quad} & & PT
\end{array}
$$

Then $\mathbf{P}$ lifts to a unique monad $\mathbf{P}^{\mathbf{T}} = (P^{\mathbf{T}}, \tilde{\mu}, \tilde{\eta})$ in $\mathcal{A}^{\mathbf{T}}$, such that $P^{\mathbf{T}}$ lifts P, and for all $N \in \mathcal{A}^T$ we have $U^{\mathbf{T}}(\tilde{\eta}_N) = (\eta^P)_{U^{\mathbf{T}}N}$ and $U^{\mathbf{T}}(\tilde{\mu}_N) = (\mu^P)_{U^{\mathbf{T}}N}$.

Indeed, such a lifting is defined by the formulas $P^{\mathbf{T}}(M, \nu) := (PM, P(\nu) \circ l_M)$, $\mu^P_{(M,\nu)} = \mu_M$, $\eta^P_{(M,\nu)} = \eta_M$. On the other hand, if $\mathbf{P}^{\mathbf{T}}$ is a lifting of $\mathbf{P}$ then a distributive law $l = \{l_M\}$ is defined, namely l_M is the composition

$$TPM \stackrel{TP(\eta_M)}{\longrightarrow} TPTM \stackrel{U^{\mathbf{T}}(\epsilon_{PF^{\mathbf{T}}M})}{\longrightarrow} PTM.$$

where $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is the free $\mathbf{T}$-module functor from **8.2**.

Distributive laws from $\mathbf{T}$ to $\mathbf{P}$ are in a canonical bijective correspondence with those monads in $\mathcal{A}$ whose underlying functor is PT, whose unit is $\eta_P \star \eta_T$, such that $P\eta^T : P \to PT$, and $\eta^P_T : T \to PT$ are triple maps and which satisfy the **middle unitary law** $\mu \circ (P(\eta^T) \star \eta^P_T) = \mathrm{id} : PT \to PT$ (cf. [10]). In this correspondence, $\mu^{PT}_M : PTPTM \to PTM$ is obtained by $\mu^{PT}_M = (\mu^P \star \mu^T) \circ P(l_{TM})$, and conversely, l_M by composition $TPM \stackrel{TP(\eta_M)}{\to} TPTM \stackrel{\eta_{TPTM}}{\to} PTPTM \stackrel{\mu^{PT}_M}{\to} PTM$.

9.1b Distributive law from a comonad $\mathbf{G} = (G, \delta^G, \epsilon^G)$ **to a comonad** $\mathbf{F} = (F, \delta^F, \epsilon^F)$ is a natural transformation $l : F \circ G \Rightarrow G \circ F$ such that

$$\begin{aligned} G(\epsilon^F) \circ l &= (\epsilon^F)_G, \quad (\delta^G)_F \circ l = G(l) \circ l_G \circ F(\delta^G), \\ (\epsilon^G)_F \circ l &= F(\epsilon^G), \quad G(\delta^F) \circ l = l_F \circ F(l) \circ (\delta^F)_G. \end{aligned}$$

9.1c Mixed distributive law from a monad $\mathbf{T}$ to a comonad $\mathbf{G}$: a natural transformation $l : TG \Rightarrow GT$ such that

$$\begin{aligned} \epsilon^T \circ l &= T(\epsilon), \quad l \circ \mu_G = G(\mu) \circ l_T \circ T(l), \\ l \circ \eta_G &= G(\eta), \quad l \circ T(\delta) = \delta_T \circ G(l) \circ l_G. \end{aligned}$$

Such an l corresponds to a lifting of the comonad $\mathbf{G}$ to a comonad $\mathbf{G}^{\mathbf{T}}$ in $\mathcal{A}^{\mathbf{T}}$, where $G^{\mathbf{T}}(M, \nu) = (MG, G(\nu) \circ l_M)$.

9.1d Mixed distributive law from a comonad $\mathbf{G}$ to a monad $\mathbf{T}$: a natural transformation $l : GT \Rightarrow TG$ such that

$$\begin{aligned} T(\epsilon) \circ l &= \epsilon_T, \quad l \circ G(\mu) = \mu_G \circ T(l) \circ l_T \\ l \circ \eta_G &= G(\eta), \quad T(\delta) \circ l = l_G \circ G(l) \circ \delta_T. \end{aligned}$$

Such distributive laws are in a correspondence with liftings of a monad $\mathbf{T}$ to a monad $\mathbf{T}^{\mathbf{G}}$ in $\mathbf{G} - \mathrm{Comod}$.

9.2 Examples are abundant ([21, 119]; [131],II). In a common scenario, the objects in a category of interest are in fact objects in a simpler 'base' category, together with multiple extra structures, satisfying "compatibility" conditions between the structures, which correspond to a fixed choice

of distributive laws. Consider, left R-modules and right S-modules, where the base category is the category **Ab** of abelian groups. The rebracketing map $(R\otimes?) \otimes S \mapsto R \otimes (? \otimes S)$ gives rise to a distributive law from $R\otimes?$ and $? \otimes S$ in **Ab**. Thus, it induces a monad **V** with underlying functor $V = (S\otimes?) \circ (T\otimes?)$. **V**-modules over U are precisely $R - S$-bimodules. Similarly, for a group (or Hopf algebra) G, one can describe G-equivariant versions of many standard categories of sheaves or modules with extra structure, by considering one (co)monad for the underlying structure and another expressing the G-action.

9.3 A monad (T, μ, η) in arbitrary 2-category (even bicategory) $\mathcal{C}$ has been studied ([131]): $T : X \to X$ is now a 1-cell, where X is a fixed 0-cell, and μ, η 2-cells should satisfy analogous axioms as in the ordinary case, which corresponds to $\mathcal{C} = \mathrm{Cat}$. On the other hand, if $\mathcal{C}$ is a bicategory with a single object X, it may be identified with a monoidal 1-category. The distributive laws in that case supply a notion of compatibility of monoids and comonoids in an arbitrary monoidal category. The distributive laws between monoids and comonoids in $\mathrm{Vec}_{\mathbf{k}}$ are called *entwining structures* ([24]).

9.4 Let $\mathbf{T} := (T, \mu, \eta)$ be a monad, and $Q^* : \mathcal{A} \to \mathcal{B}$ a localization functor. The monad $\mathbf{T}$ is **compatible** with the localization if its underlying endofunctor T is compatible with the localization, i.e. there is a functor $T_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ with $Q^*T = T_{\mathcal{B}}Q^*$, cf. **3.5**. In that case, $T_{\mathcal{B}}$ is the underlying endofunctor of a unique monad $\mathbf{T}_{\mathcal{B}} := (T_{\mathcal{B}}, \mu^{\mathcal{B}}, \eta^{\mathcal{B}})$ in $\mathcal{B}$ such that $(\mu^{\mathcal{B}})_{Q^*N} = Q^*(\mu_N)$ for every N in $\mathrm{Ob}\,\mathcal{A}$.

Proof. Let $f : N \to N'$ be a morphism in $\mathcal{A}$, and $g : Q^*N' \to M$ an isomorphism in $\mathcal{B}$. Consider the diagram

$$
\begin{array}{ccccccc}
T_{\mathcal{B}}T_{\mathcal{B}}Q^*N & \xrightarrow{=} & Q^*TTN & \xrightarrow{Q^*\mu_N} & Q^*TN & \xrightarrow{=} & T_{\mathcal{B}}Q^*N \\
\downarrow{\scriptstyle T_{\mathcal{B}}T_{\mathcal{B}}Q^*f} & & \downarrow{\scriptstyle Q^*TTf} & & \downarrow{\scriptstyle Q^*Tf} & & \downarrow{\scriptstyle T_{\mathcal{B}}Q^*f} \\
T_{\mathcal{B}}T_{\mathcal{B}}Q^*N' & \xrightarrow{=} & Q^*TTN' & \xrightarrow{Q^*\mu_{N'}} & Q^*TN' & \xrightarrow{=} & T_{\mathcal{B}}Q^*N' \\
\downarrow{\scriptstyle T_{\mathcal{B}}T_{\mathcal{B}}g} & & & & & & \downarrow{\scriptstyle T_{\mathcal{B}}g} \\
T_{\mathcal{B}}T_{\mathcal{B}}M & & & \xrightarrow{\quad\quad (\mu^{\mathcal{B}})_M \quad\quad} & & & T_{\mathcal{B}}M.
\end{array}
$$

The upper part of the diagram clearly commutes. In particular, if Q^*f is identity, then $Q^*\mu_N = Q^*\mu_{N'}$. The vertical arrows in the bottom part are isomorphisms, so there is a map $(\mu^{\mathcal{B}})_M$ filling the bottom line. One has to show that this map does not depend on the choices and that such maps form a natural transformation. The localization functor is a composition of a quotient functor onto the quotient category and an equivalence. We

may assume that Q^* is the functor onto the quotient category. Then, by the construction of the quotient category, every morphism g is of the zig-zag form as a composition of the maps of the form Q^*f and formal inverses of such maps, and if g is an isomorphism, both kinds of ingredients are separately invertible in $\mathcal{B}$. To show that $(\mu^B)_M = (T_\mathcal{B}g) \circ Q^*(\mu_{N'}) \circ (T_\mathcal{B}T_\mathcal{B}g)^{-1}$ for every isomorphism $g : Q^*N' \to M$ is the consistent choice, we use the upper part of the diagram repeatedly (induction by the length of zig-zag) for the zig-zag isomorphism $h = (g_1)^{-1}g_2 : Q^*N_2 \to Q^*N_1$ where $g_i : Q^*N_i \to M$. One obtains $T_\mathcal{B}h \circ Q^*\mu_{N_2} \circ (T_\mathcal{B}T_\mathcal{B}h)^{-1} = Q^*\mu_{N_1}$, hence $(T_\mathcal{B}g_2) \circ Q^*(\mu_{N_2}) \circ (T_\mathcal{B}T_\mathcal{B}g_2)^{-1} = (T_\mathcal{B}g_1) \circ Q^*(\mu_{N_1}) \circ (T_\mathcal{B}T_\mathcal{B}g_1)^{-1}$.

Upper part of the diagram also shows the naturality for $\mu^\mathcal{B}$ with respect to each arrow of the form Q^*f and with respect to formal inverses of such. For any morphism $h : M \to M'$ in $\mathcal{B}$, using its zig-zag representation, we extend this to the naturality diagram $(\mu^\mathcal{B})_{M'} \circ T_\mathcal{B}T_\mathcal{B}h = T_\mathcal{B}T_\mathcal{B}h \circ (\mu^\mathcal{B})_M$. Uniqueness of $\mu^\mathcal{B}$ is clear by the requirement $(\mu^\mathcal{B})_{Q^*N} = Q^*(\mu_N)$ and the naturality. The unit morphism $\eta : 1_\mathcal{B} \to T_\mathcal{B}$ satisfies $g \circ \eta_M^\mathcal{B} = T_\mathcal{B}(g) \circ Q^*(\eta_N)$ for every isomorphism $g : Q^*N \to M$ in $\mathcal{B}$ such that $N \in \mathrm{Ob}\,\mathcal{A}$. In particular, $(\eta^\mathcal{B})_{Q^*N} = Q^*(\eta_N)$. The very axioms of a monad may be checked in a similar vein.

9.4b If $\mathbf{G} = (G, \delta, \epsilon)$ is a comonad and the endofunctor G is compatible with each localization in family $\{Q_\lambda^* : \mathcal{A} \to \mathcal{B}_\lambda\}_{\lambda\in\Lambda}$, then there is a unique family of comonads $\{\mathbf{G}_\lambda = (G_\lambda, \delta_\lambda, \epsilon_\lambda)\}_{\lambda\in\Lambda}$ such that $Q_\lambda^*G = G_\lambda Q_\lambda^*$ for each λ. We have then $(\delta_\lambda)_{Q_\lambda^*M} = Q_\lambda^*(\delta_M)$ and $(\epsilon_\lambda)_{Q_\lambda^*M} = Q_\lambda^*(\epsilon_M)$ for every $M \in \mathrm{Ob}\,\mathcal{A}$.

9.5 If $Q^* : \mathcal{A} \to \mathcal{B}$ is a localization with right adjoint Q_*, and T' is an endofunctor in $\mathcal{B}$, then $Q_*T'Q^*$ is compatible with Q^*. Indeed ϵ is an isomorphism by **3.4**, hence $\epsilon_{T'Q^*} : Q^*Q_*T'Q^* \Rightarrow T'Q^*$ is an isomorphism, and the assertion follows by **3.5**.

9.6 Example from Hopf algebra theory. Let B be a $\mathbf{k}$-bialgebra and (E, ρ) a right B-comodule together with a multiplication $\mu : E \otimes_\mathbf{k} E \to E$ making it a B-comodule algebra, i.e. an algebra in the category of right B-comodules. The coaction $\rho : E \to E \otimes H$ is **compatible** with a fixed Ore localization $\iota : E \to S^{-1}E$ if there is a coaction $\rho_S : S^{-1}E \to S^{-1}E \otimes H$ which is an algebra map and such that $\rho_S \circ \iota = (\iota \otimes \mathrm{id}_B) \circ \rho$. B induces a natural comonad $T = T(B, E)$ in $E - \mathrm{Mod}$, such that $(E - \mathrm{Mod})^T$ is a category of so-called (E, B)-Hopf modules. The compatibility above ensures that the localization lifts to a localization of $(E - \mathrm{Mod})^T$ ([122], 8.5). Hence, T is compatible with the localization in the usual sense, with numerous applications of this type of situation ([118, 120, 121, 122]).

9.7 The compatibility of certain localizations of noncommutative spaces with differential functors is central in the treatment [79, 80, 81, 82] of $\mathcal{D}$-modules on noncommutative spaces.

9.8 Distributive laws become a much simpler issue when both monads in question are idempotent in the sense of **8.9**. In the localization literature this is roughly the situation treated under the name of **mutual compatibility of localizations**.

Let S, W be two Ore sets. Then the set SW of products $\{sw | s \in S, w \in W\}$ is not necessarily multiplicatively closed.

Suppose SW is multiplicatively closed. This means that $\forall s \in S, \forall w \in W$ if the product ws is in SW then $\exists w' \in W, \exists s' \in S$ such that $ws = s'w'$. Suppose now $M := {}_R M \in R-\mathrm{Mod}$. Each element in $S^{-1}R \otimes_R W^{-1}R \otimes_R M$ is of the form $s^{-1} \otimes w^{-1} \otimes m$ with $s \in S$, $w \in W$ and $m \in M$. By a symmetric argument, $ws \in SW$, and thus $\exists w' \in W$, $\exists s' \in S$ such that $ws = s'w'$. Choosing s', w' by this rule we obtain an assignment $s^{-1} \otimes w^{-1} \otimes m \mapsto (w')^{-1} \otimes (s')^{-1} \otimes m$. We claim that this assignment is well-defined and a map of left R-modules $S^{-1}R \otimes_R W^{-1}R \otimes_R M \to W^{-1}R \otimes_R S^{-1}R \otimes_R M$. As M runs through $R - \mathrm{Mod}$, such maps form a natural transformation $Q_S Q_W \to Q_W Q_S$ of functors, which is clearly an isomorphism.

In fact, this natural transformation is a distributive law. Although the compatibility of Q_S and Q_W is symmetric, the converse does not hold: compatibility does not mean that SW is multiplicatively closed. Indeed, let R be a $\mathbb{C}$-algebra with two generators a, b and relation $ab = qba$ where $q \neq \pm 1, 0$. Then the set multiplicatively generated by A and set multiplicatively generated by B are 2-sided Ore sets, and the corresponding localization functors are compatible; however AB is not multiplicatively closed.

If S, W are left Ore in R, that does not mean that $\iota_W(S)$ is left Ore in $W^{-1}R$. Namely, the left Ore condition for (the image of) S in $W^{-1}R$ includes the following: $\forall s \in S, \forall t \in T, \exists s' \in S, \exists (w')^{-1}r' \in W^{-1}R, (w')^{-1}r's = s'w^{-1}$. If R is a domain, this means that $r'sw = w's'$. This is almost the same condition as that SW is multiplicatively closed (above), except that one can choose extra r'. In the same away as in the former case, we derive the compatibility of Q_S and Q_W. If we change left Ore sets to right Ore sets, or S being Ore in $W^{-1}R$ to W being Ore in $S^{-1}R$ we get similar "Ore" equations $swr = w's'$, $wsr = s'w'$ etc. From the abstract point of view (say torsion theories) these compatibilities are indistinguishable.

The compatibility implies that the localization at the smallest multiplicative set generated by S and W is isomorphic to the consecutive localization by S and then by W. This simplifies the formalism of localization

(cf. semiseparated schemes and Čech resolutions of [109], cf. **8.8**, [62] etc.).

10   Commutative localization

Here we describe specifics in the commutative case, and further motivation from commutative algebraic geometry, and its abstractions.

10.1 Suppose R is a unital associative ring, $Z(R)$ its center, and $S \subset Z(R)$ a multiplicative subset. Obviously, S is automatically a left and right Ore subset in R, with simpler proofs for the construction and usage of the Ore localization. We say that $S^{-1}R = S \times R/ \sim$ is the **commutative localization** of R at S. *The equivalence relation $\sim$ (**4.13**) simplifies to*

$$\boxed{s^{-1}r \sim s'^{-1}r' \Leftrightarrow \exists \tilde{s} \in S, \; \tilde{s}(sr' - s'r) = 0.}\tag{13}$$

Proof. By the definition, $\exists \tilde{s} \in S$, $\exists \tilde{r} \in R$, such that $\tilde{s}r = \tilde{r}r'$ and $\tilde{s}s = \tilde{r}s'$. Therefore,

$$\tilde{s}sr' = \tilde{r}s'r' = \tilde{r}r's' = \tilde{s}rs' = \tilde{s}s'r.$$

Unlike sometimes (mis)stated in the literature (e.g. [138],p.14), the commutative formula (13) (and variants of it) is inappropriate even for mildly noncommutative rings and even 2-sided Ore sets which are not in center. E.g. take the unital $\mathbb{C}$-algebra generated by two elements b and d with $bd = qdb$, where $\mathbb{C} \ni q \neq 1$. That algebra has no zero divisors. Let S be the 2-sided Ore set multiplicatively generated by b and d. Formula $b^{-1} = (db)^{-1}d$, and the criterion above would imply that $db = bd$ with contradiction.

For general R and S, formula (13) is actually not even an equivalence relation on $R \times S$.

10.2   General requirements on scheme-like theories. One wants to mimic several major points from the classical case. We first decide which geometric objects constitute the category $\mathcal{C}$ of affine schemes; then find a suitable larger geometric category $\mathfrak{Esp}$ of spaces, in the sense that it is equipped with a fully faithful functor $\mathcal{C} \hookrightarrow \mathfrak{Esp}$, where the objects in the image will be called here *geometric affine schemes*; finally there is a gluing procedure which assigns to a collection $\{\mathfrak{C}_h\}_{h \in H}$ of geometric affine schemes and some additional 'gluing' data Z, a space which may be symbolically denoted by $(\coprod_h \mathfrak{C}_h)/Z \in \mathfrak{Esp}$, together with canonical morphisms $\mathfrak{C}_h \to (\coprod_h \mathfrak{C}_h)/Z$ in $\mathfrak{Esp}$. For a fixed (type of) gluing procedure $\mathcal{G}$, and including all the isomorphic objects, one constructs this way a subcategory of **locally affine spaces** of type $(\mathcal{C}, \mathcal{G})$ in $\mathfrak{Esp}$.

Additional requirements and typical choices are in place.

10.2a Abstract affine schemes. Most often one deals with some monoidal category $\tilde{\mathcal{A}} = (\mathcal{A}, \otimes, 1_{\tilde{A}})$. Then $\mathcal{C} = \mathrm{Aff}(\tilde{A}) := (\mathrm{Alg}(\tilde{A}))^{\mathrm{opp}}$ is the **category of affine schemes in** $\mathcal{A}$ i.e. the *opposite to the category of algebras (monoids) in* $\tilde{A}$. The basic example is the monoidal category of R-bimodules, where R is a **k**-algebra over a commutative ring **k**. The affine schemes in this category are given by **k**-algebra maps $f : R \to R'$ (making such R' an R-bimodule).

If $\mathcal{A}$ is *symmetric* via a symmetry τ where $\tau_{AB} : A \otimes B \to B \otimes A$, then one may consider only τ-commutative algebras (A, μ, η), i.e. for which $\mu \circ \tau = \mu$. τ-**commutative affine schemes** in $\mathcal{A}$ are the objects of the opposite of the category of τ-commutative algebras $\mathcal{C} = \mathrm{cAff}(\tilde{A}, \tau) := (\mathrm{cAlg}(\tilde{A}, \tau))^{\mathrm{opp}}$ (one often skips τ in the notation). Examples are (super)commutative affine schemes in a $\otimes$-category of **k**-modules and also the opposite to the category $\underline{\mathrm{cdga}}_{\mathbf{k}}$ of commutative differential graded **k**-algebras, which is important in recent 'derived algebraic geometry' program ([132, 133]).

10.2b Gluing for ringed topological spaces (and a version for local (l.) rings). A (locally) ringed space is a pair (X, F) consisting of a topological space X, and a ('structure') sheaf F of (l.) rings on X. A morphism of (l.) ringed spaces is a continuous map $f : X \to X'$ with a *comorphism* i.e. a map of sheaves of (l.) rings $F' \to f^* X'$ over X'. We obtain a category $\mathfrak{rSp}$ ($\mathfrak{lSp}$). Given a full subcategory $\mathcal{A}$ of $\mathfrak{rSp}$, considered as a category of (heuristic term here) "local models" we may consider all (l.) ringed space X for which there is a cover (in usual sense) $X = \cup_\alpha X_\alpha$ of underlying topological spaces and for each α an isomorphism $X_\alpha^0 \cong X_\alpha$ in $\mathfrak{rSp}$ ($\mathfrak{lSp}$) where X_α^0 is in $\mathcal{A}$. More abstractly, but equivalently, consider all families of morphisms $\{i_\alpha : X_\alpha \to X\}_\alpha$, which are 'covers by embeddings': topologically covers of X by families of monomorphisms (continuous, open and injective), and sheaf-wise isomorphisms on stalks. The intersections $X_\alpha \cap X_\beta$ represent fibred products $X_\alpha \times_X X_\beta$ in $\mathfrak{rSp}$. Spaces glued from objects in $\mathcal{A}$ are nothing but the colimits of the diagrams of the type $\coprod_{\alpha\beta} Y_{\alpha\beta} \rightrightarrows \coprod Y_\alpha$ where each of the morphisms $Y_{\alpha\beta} \to Y_\alpha$ and $Y_{\alpha\beta} \to Y_\beta$ are embeddings, and $Y_\beta \in \mathrm{Ob}\,\mathcal{A}$. There is a natural condition on $\mathcal{A}$: each X in $\mathcal{A}$ as a topological space has a basis of topology made out of (spaces of) some family of objects in $\mathcal{A}$ (or isomorphic to them); and this family may be chosen so that the restrictions of the structure sheaf agree. In the theory of schemes, affine schemes are such models: affine subschemes make a basis of topology, but not every open subset is affine; nor their intersections. Still the intersections and colimits exist from the start, in our ambient category of ringed spaces which is big enough.

10.2c Grothendieck (pre)topologies (G.(p)t.) and gluing in commutative algebraic geometry ([85, 142]). Schemes are glued from affine schemes in the Zariski topology (which may be considered both as an ordinary topology and a G.(p)t.); useful generalizations (e.g. algebraic spaces) in flat and étale G.(p)t. etc. A **sieve** is an assignment of a collection $J(R)$ of morphisms in $\mathcal{C}$ ([85]) with target R, such that if the target of f is $J(R)$ and $g : R \to R'$ then $g \circ f \in J(R')$. A G.t. is a collection of sieves $J(R)$ with target R for each R in $\mathrm{Ob}\,\mathcal{C}$, such that if $f \in J(R)$ and $g : R \to R'$ then $g \circ f \in J(R'))$, satisfying some axioms ([85]). A G.pt. in a category $\mathcal{C}$ with fibred products is a class $\mathcal{T}$ of families $\{U_\alpha \to U\}_\alpha$ of morphisms (with one target per family), such that $\{\mathrm{Id} : U \to U\} \in \mathcal{T}$; if $\{f_\alpha : U_\alpha \to U\}_\alpha \in \mathcal{T}$ and $\forall \alpha\, \{g_{\alpha\beta} : U_{\alpha\beta} \to U_\alpha\}_\beta \in \mathcal{T}$ then $\{f_\alpha \circ g_{\alpha\beta}\}_{\alpha,\beta} \in \mathcal{T}$; and finally if $\{f_\alpha : U_\alpha \to U\}_\alpha \in \mathcal{T}$ and $g : V \to U$ is a morphism then $\{g^*(f_\alpha) : V \times_U U_\alpha \to V\} \in \mathcal{T}$. Elements of $\mathcal{T}$ are called **covers** (in $\mathcal{T}$), and the pair $(\mathcal{C}, \mathcal{T})$ a site. To any (ordinary) topological space X one associates a "small" site $\mathfrak{Ouv}_X$: objects are open subsets in X; morphisms are inclusions; $\{U_\alpha \hookrightarrow U\}_\alpha$ is a cover if $\cup_\alpha U_\alpha = U$.

A presheaf F of sets on X is a functor $F : (\mathfrak{Ouv}_X)^{\mathrm{op}} \to \mathrm{Sets}$; a presheaf F on any site $(\mathcal{C}, \mathcal{T})$ with values in a category $\mathcal{D}$ with products is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Given a cover $\{U_\alpha \to U\}_\alpha \in \mathcal{T}$, there are two obvious embeddings $\prod U_\alpha \times_U U_\beta \rightrightarrows \prod_\alpha U_\alpha$. A presheaf is a sheaf on $(\mathcal{C}, \mathcal{T})$ if $\{U_\alpha \to U\}_\alpha \in \mathcal{T}$ if for every such cover the induced diagram $F(U) \longrightarrow \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha\beta} F(U_\alpha \times_U U_\beta)$ is an equalizer diagram.

For gluing, one again needs some bigger ambient category (or, instead, some universal construction). Our local models are now (commutative) affine schemes $\mathrm{Aff} := \mathrm{Aff}(\mathrm{Ab})$ with a G.pt. $\mathcal{T}$. The Yoneda embedding $X \mapsto X^\vee := \mathrm{Aff}(?, X)$ is a fully faithful functor from Aff into the category $\mathrm{PFas}\,(\mathrm{Aff})$ of presheaves of sets on Aff. One typically deals with *subcanonical G.t.* which means that the presheaves in the Yoneda image (representable functors) are sheaves. As in the case of ordinary topologies, to construct the global locally $\mathcal{T}$-affine spaces, one needs colimits of certain diagrams of the form $\prod_{\alpha\beta} V_{\alpha\beta} \rightrightarrows \prod_\alpha U_\alpha$, where, in the addition, the colimit cone $\prod U_\alpha \to U$ corresponds to a cover in $\mathcal{T}$. As for example, the nonseparated schemes in Zariski topology, some locally $\mathcal{T}$-affine spaces may not be produced this way with $V_{\alpha\beta}$ being in Aff. Similar problems for biflat covers by localizations in noncommutative geometry are known (lack of compatibility of localizations; nonsemiseparated covers). Furthermore, one needs to extend the notion of $\mathcal{T}$-covers to the target category of sheaves. We hope that the reader sees at this point the meaning of this abstract machinery. We

won't proceed with the full construction. Namely, in the commutative case, it is usually replaced by equivalent constructions. For example, to construct the algebraic spaces, one usually does not glue affine schemes "over intersections" but rather starts with an equivalence relation in the category of all separated schemes. In noncommutative case, G.t. are elaborated in [69, 110] and partly in [115]. There is also an approach to G.t. and quasicoherent sheaves for noncommutative affine schemes by ORLOV ([99]), utilizing ringed sites and sheaves of groupoids, but implicit application to the construction of noncommutative locally affine spaces is not given.

10.3 Though we assume that the reader has been exposed to the commutative scheme theory, we here sketch the basic construction of an affine scheme, as a part of a widely-known easy generalization to pairs of the form (noncommutative ring, central subring), cf. [106].

A left ideal $\mathfrak{p} \subset R$ is **prime** if for any two left ideals I, J, if $IJ \subset \mathfrak{p}$ then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$. A left ideal $\mathfrak{p} \subset R$ is **completely prime** if $fg \in \mathfrak{p}$ then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$; equivalently $R/\mathfrak{p}$ is a domain; or $R\backslash\mathfrak{p}$ is multiplicative. Each completely prime ideal is prime: otherwise one could find $f \in I\backslash\mathfrak{p}$ and $g \in J\backslash\mathfrak{p}$, such that $fg \in IJ \subset \mathfrak{p}$ with contradiction. If R is commutative the converse holds, as one can see by specializing the definitions to the principal left ideals $I = Rf$, $J = Rg$, $IJ = RfRg = Rfg$.

Consider the category $\mathcal{R}$, whose objects are pairs (R, C), of a unital ring R and a central subring $C \subset Z(R)$; and morphisms $(R, C) \to (R', C')$ are maps of rings $\phi : R \to R'$ such that $\phi(C) \subset C'$.

$\operatorname{Spec} C$ is the set of all prime ideals $\mathfrak{p}$ of C. For any ideal $I \subset C$, define $V(I) \subset \operatorname{Spec} C$ as the set of all $\mathfrak{p} \subset C$, such that $I \subset \mathfrak{p}$. Sets of the form $V(I)$ depend only (contravariantly with respect to inclusions) on the radical $\sqrt{I}$ (the intersection of all prime ideals containing I) and satisfy the axioms of antitopology. Complements of such sets hence form a topology on $\operatorname{Spec} C$, called **Zariski topology**. **Principal open sets** are the sets of the form $U_f = V((f))$, where (f) is the (principal) ideal generated by $f \in C$. They make a basis of Zariski topology, i.e., any Zariski open set is a union of sets of that form. Open sets and inclusions form category $\mathfrak{Ouv}_C$. Principal open sets and inclusions form its full subcategory $\mathfrak{Ouv}\mathfrak{P}_C$.

Define $\mathcal{O}'_C(U_f) := C[f^{-1}]$ and $\mathcal{O}'_{R,C}(U_f) := R[f^{-1}]$. Every inclusion $U_f \hookrightarrow U_g$ induces the unital ring maps $\phi_{f,g,i} : \mathcal{O}'_i(U_g) \to \mathcal{O}'_i(U_f)$, $i \in \{C, (R,C)\}$. Hence we have contravariant functors $\mathcal{O}'_i : \mathfrak{Ouv}\mathfrak{P}_C \to \underline{Rings}$. Natural inclusions $\mathbf{in}_f : C[f^{-1}] \to R[f^{-1}]$ form a natural transformation, i.e., $\mathcal{O}'_C$ is a subfunctor of $\mathcal{O}'_{C,R}$. Functors $\mathcal{O}'_i$ extend naturally to functors $\mathcal{O}_i : \mathfrak{Ouv}_C \to \underline{Rings}$ which are sheaves, and this requirement fixes sheaves

$\mathcal{O}_i$ uniquely up to isomorphism of sheaves. Namely, represent any open set U as the union $\cup U_f$ of a family $\mathcal{U}$ of (some or all) $U_f \subset U$. Define a diagram $\Delta(\mathcal{U})$ in $\underline{Rings}$ as follows. Vertices are of the form $\mathcal{O}_i(U_f)$ and of the form $\mathcal{O}_i(U_{fg}) = \mathcal{O}_i(U_f \cap U_g)$ where $U_f, U_g \in \mathcal{U}$, and the arrows are $\phi_{f,fg,i}$. Then $\mathcal{O}_i(U)$ has to be isomorphic to the inverse limit of the diagram $\Delta(\mathcal{U})$ and may be consistently set so. Moreover the natural transformation **in** extends making $\mathcal{O}_C$ a subfunctor of $\mathcal{O}_{C,R}$. In fact, $\mathcal{O}_{C,R}$ is an algebra in the monoidal category of (2-sided) $\mathcal{O}_C$-modules. All the stalks $(\mathcal{O}_C)_{\mathfrak{p}}$ of $\mathcal{O}_C$ are local rings, namely the localizations $C_{\mathfrak{p}} := C[\{f^{-1}\}_{f \notin \mathfrak{p}}]$ 'at prime ideal $\mathfrak{p}$'.

Let $\psi : R \to R'$ be a map of unital rings. The inverse image $\psi^{-1}(\mathfrak{p})$ of a completely prime ideal is completely prime. Let $\psi : (R, C) \to (R', C')$ be a morphism in $\mathcal{R}$, and let map $\phi : C \to C'$ agrees with ψ on C. One has map $\phi^* : \operatorname{Spec} C' \to \operatorname{Spec} C$ given by $\phi^* : \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$. If $U \subset \operatorname{Spec} C'$ is open, then $\phi^{-1}(U) \subset \operatorname{Spec} C$ is open as well, because $\phi^*(V(I)) = V(\phi^{-1}(I))$ for each ideal $I \subset C'$. Hence, ϕ^* is continuous. If $g \notin \phi^{-1}(\mathfrak{p})$ then $\phi(g) \notin \mathfrak{p}$. Thus all elements $g \in C$, newly inverted in $C_{\phi^{-1}(\mathfrak{p})}$ (and $R_{\phi^{-1}(\mathfrak{p})}$) are also invertible in $C'_{\mathfrak{p}}$ (and $R'_{\mathfrak{p}}$). Hence, by the universality of localization, one has a unique map $\psi_{\mathfrak{p}} : (R_{\phi^{-1}(\mathfrak{p})}, C_{\phi^{-1}(\mathfrak{p})}) \to (R'_{\mathfrak{p}}, C'_{\mathfrak{p}})$ such that $\psi_{\mathfrak{p}} \circ \iota_{\phi^{-1}(\mathfrak{p})} = \iota_{\mathfrak{p}} \circ \psi$. Define $\phi_U : \mathcal{O}_{R,C}(\phi^* U) \to \mathcal{O}_{R',C'}(U)$ by $(\phi_U(r))_{\mathfrak{p}} = \psi_{\mathfrak{p}}(r_{\phi^{-1}(\mathfrak{p})})$. One has to check that $\phi_U(r)$ is indeed in $\mathcal{O}_{R',C'}(U)$, i.e., that $\mathfrak{p} \mapsto \psi_{\mathfrak{p}}(r_{\phi^{-1}(\mathfrak{p})})$ is indeed a section. For this consider $U_f \subset U$ affine, i.e., $U_f = \{\mathfrak{p} \,|\, f \notin \mathfrak{p}\}$ and $\phi^* U_f := \{\phi^{-1}(\mathfrak{p}), \mathfrak{p} \not\ni f\}$. An argument as above gives map $\psi_f : \mathcal{O}_{R,C}(\phi^* U_f) \to \mathcal{O}_{R',C'}(U_f)$ satisfying $\psi_f \circ \iota_{\phi^{-1}(U_f)} = \iota_{U_f} \circ \psi$. It is easy to check then that ψ_f induces $\psi_{\mathfrak{p}}$ in stalk over $\mathfrak{p} \in U_f$. As a result, we obtain a map $\psi^{\sharp} : \phi_* \mathcal{O}_{R,C} \to \mathcal{O}_{R',C'}$ of sheaves over $\operatorname{Spec} C'$.

10.3a Let $\mathbb{lSp}_2$ be a category whose object are locally ringed spaces $(X, \mathcal{O})$ in $\mathbb{lSp}$ together with a sheaf $\mathcal{O}^{\mathrm{nc}}$ of noncommutative algebras in the category of $\mathcal{O}$-modules. A morphism $\psi : (X, \mathcal{O}_X, \mathcal{O}_X^{\mathrm{nc}}) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^{\mathrm{nc}})$ in $\mathbb{lSp}_2$ is a morphism $\psi^{\mathrm{c}} : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathbb{lSp}$, together with a map of sheaves of $\mathcal{O}_X$-modules $\psi^{\sharp} : \mathcal{O}_Y^{\mathrm{nc}} \to \psi^* \mathcal{O}_X^{\mathrm{nc}}$.

We have above constructed a functor $\operatorname{Spec}_2 : \mathcal{R} \to \mathbb{lSp}_2$.

10.3b In fact, $\mathcal{O}^{\mathrm{nc}}$ is in a smaller category of *quasicoherent sheaves of $\mathcal{O}$-modules* (shortly: quasicoherent modules). A presheaf F of $\mathcal{O}$-modules on a ringed space $(X, \mathcal{O})$ is quasicoherent (EGA 0.5.1.1) if $\forall x \in X \; \exists U^{\mathrm{open}} \ni x$ with an exact sequence $\mathcal{O}^I \to \mathcal{O}^J \to F \to 0$ where $\mathcal{O}^I, \mathcal{O}^J$ are free $\mathcal{O}$-modules (of possibly infinite rank). If the ringed space is locally $\mathcal{T}$-affine for some Grothendieck topology $\mathcal{T}$ on the category of commutative affine schemes, then one may take U affine, and an equivalent definition of quasi-

coherence is that for any pair of affine open subsets $W \subset V \subset X$,

$$F(W) = \mathcal{O}(W) \otimes_{\mathcal{O}(V)} F(V). \tag{14}$$

The noncommutative structure sheaf $\mathcal{O}^{\mathrm{nc}}$ of $\mathrm{Spec}_2(R, C)$ is a quasicoherent $\mathcal{O}$-module. For presheaves of $\mathcal{O}^{\mathrm{nc}}$ left modules one may use the same formula (14). For bimodules one distinguishes left and right quasicoherence [67, 78, 147] (in the right hand version the tensoring with $\mathcal{O}(W)$ in formula (14) is from the RHS instead). Formula (14) is nothing else but the formula for a localization functor $Q^* : F(V) \mapsto F(W)$ from $\mathcal{O}(V)$-modules to $\mathcal{O}(W)$-modules. In this case, both Q^* and its right adjoint Q_* are exact functors. Ring theorists call such localizations of full module categories **perfect localizations** ([62]). Equivalently, the canonical forgetful functor from the localized category to the category of modules over the localized ring is an equivalence of categories.

F. VAN OYSTAEYEN ([138]) defines quasicoherent presheaves on the lattice of hereditary torsion theories (= localizations where Q^* is exact and the torsion subcategory is coreflective) over a $\mathbb{Z}_{\geq 0}$-graded rings, by using the appropriate localization instead of tensoring. For affine case see [113], I 6.0.3 (b) and I 6.2. One has theorems on gluing of such modules using Barr-Beck lemma.

D. ORLOV ([99]) defines quasicoherent (pre)sheaves on X where X is a presheaf of sets (in particular, by Yoneda, on any object) on a given ringed site $(\mathcal{C}, \mathcal{T})$.

10.4 Now we will quote two theorems. A theorem of DELIGNE shows that in the context of commutative affine schemes, a formula which can be recognized as a localization at a Gabriel filter (see below), describes the behaviour of the category of quasicoherent sheaves under passage to a not necessarily affine subset. Hence the "noncommutative localization" is already seen there! The theorem may be proved directly, and we suggest to the reader to at least convince oneself that the RHS formula is indeed an A-module. "Deligne theorem" ([59]) in fact, cf. ([60], Appendix)[2], has been originally inferred from the Gabriel theorem below – the general statement that quasicompact open subsets of (nonaffine) schemes correspond always to exact localizations of abelian categories; combined with the Gabriel's formulas on how such localizations look like ([42]).

10.4a Deligne's theorem. Let $X = \mathbf{Spec}\,A$ be an affine Noetherian scheme, i.e. A is a commutative Noetherian ring. Let U be a Zariski open set (not necessarily affine!), and I an ideal such that $V(I) = X \backslash U$. Let

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[2]I thank Prof. HARTSHORNE for an email on the history of the result.

M be an A-module, and $\tilde{M}$ the corresponding quasicoherent $\mathcal{O}_X$-module. Denote by $\tilde{M}|_U$ its restriction to U. Then

$$\Gamma(\tilde{M}|_U) \cong \lim_{n\to\infty} \mathrm{Hom}_A(I^n, M).$$

This isomorphism of A-modules is natural in M.

10.4b (P. GABRIEL [42], VI.3) Let $(X, \mathcal{O}_X)$ be a scheme, and U an open subset of X, such that the canonical injection $j : U \hookrightarrow X$ is quasicompact. Let $Q_U^* : \mathfrak{Qcoh}_X \to \mathfrak{Qcoh}_U$ be a functor which associates to every quasicoherent $\mathcal{O}_X$-module M the restriction $M|_U$ of M to subscheme $(U, \mathcal{O}_X|_U)$. Functor Q_U^* canonically decomposes as $\mathfrak{Qcoh}_X \to \mathfrak{Qcoh}_X/\mathrm{Ker}\, Q_U^* \overset{\cong}{\to} \mathfrak{Qcoh}_U$ into the canonical projection onto the quotient category by $\mathrm{Ker}\, Q_U^*$ and an isomorphism.

11 Ring maps vs. module functors

11.1 As usual, we write $_RM$ when we want to emphasize that M is (understood as) a left R-module; $R - \mathrm{Mod}$ is the category of left R-modules. Let $f : R \to S$ be any map of (not necessarily unital) rings. We have the following functors induces by map f:

- **extension of scalars** $f^* : R - \mathrm{Mod} \to S - \mathrm{Mod}$, $M \mapsto S \otimes_R M$;

- **restriction of scalars** (forgetful functor) $f_* : S - \mathrm{Mod} \to R - \mathrm{Mod}$, $_SM \mapsto {}_RM$;

- $f^! : R - \mathrm{Mod} \to S - \mathrm{Mod}$, $M \mapsto \mathrm{Hom}_R({}_RS, M)$.

Denote $F \dashv G$ when functor F is left adjoint to functor G. Easy fact: $f^* \dashv f_* \dashv f^!$. Hence f^* is left exact, $f^!$ right exact and f_* exact.

11.2 It is of uttermost importance to have in mind the geometrical picture of this situation in the case when R and S are commutative and unital. Denote by $\mathfrak{lSp}$ the category of locally ringed spaces. An object $(X, \mathcal{O}_X) \in \mathfrak{lSp}$ is a pair of a topological space X and a sheaf of commutative local rings $\mathcal{O}_X$ over X; and a morphism is a pair $(f^o, f^\sharp)$ where $f^o : X \to Y$ is a map of topological spaces and a 'comorphism' $f^\sharp : f^\bullet \mathcal{O}_Y \to \mathcal{O}_X$ is a map of sheaves of local rings over X, and $f^\bullet$ a sheaf-theoretic pullback functor. There is a contravariant functor $\mathrm{Spec} : \underline{CommRings} \to \mathfrak{lSp}$ assigning spectrum to a ring. Map $f : R \to S$ is replaced by a map of locally ringed spaces

$$\mathrm{Spec}\, f = (f^0, f^\sharp) : \mathrm{Spec}\, S \to \mathrm{Spec}\, R.$$

The comorphism $f^\sharp$ is important: e.g. if $f^o : X \hookrightarrow Y$ is an inclusion, the difference between a subvariety $(X, \mathcal{O}_X)$ and, say, its n-th infinitesimal neighborhood in $(Y, \mathcal{O}_Y)$, may be expressed by a proper choice of $f^\sharp$.

11.3 After geometrizing rings, one may proceed to geometrize modules. The basic fact here is the affine SERRE's theorem establishing a correspondence $M \leftrightarrow \tilde{M}$ between the R-modules and quasicoherent sheaves of $\mathcal{O}_X$-modules, for $X = \operatorname{Spec} R$. This correspondence is an equivalence of categories $R - \mathrm{Mod} \leftrightarrow \mathfrak{Qcoh}X$. Using this equivalence of categories, functors f^* and f_* may be rephrased as additive functors

$$f^* : \mathfrak{Qcoh}X \to \mathfrak{Qcoh}Y, \quad f_* : \mathfrak{Qcoh}Y \to \mathfrak{Qcoh}X;$$

and moreover, these can be defined for any morphism $f = (f^o, f^\sharp)$ between locally ringed spaces. In this wider context, functor f^* is called the **inverse image functor** of f, given by $f^*\mathcal{F} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} f^\bullet \mathcal{F}$ where $f^\bullet \mathcal{F}$ is usual, sheaf theoretic, pullback of sheaf $\mathcal{F}$ via f^o. The restriction of scalars functor then generalizes to the **direct image functor** for sheaves which is on presheaf level given by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)),$$

and which sends $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules via $\mathcal{O}_Y$-action given by the composition of $f^\sharp \times \mathrm{id}$ and the $\mathcal{O}_X$-action. Functors f^* and f_* are naturally defined between the bigger categories, $\mathcal{O}_X - \mathrm{Mod}$ and $\mathcal{O}_Y - \mathrm{Mod}$, where they simply preserve the quasicoherence.

11.4 Functor f_* (in all settings above) is a right adjoint to f^*, hence it is left exact and the inverse image functor f^* is right exact. This suggests that a pair of adjoint additive functors between abelian categories may be viewed as (coming from) a morphism in geometrical sense. Actually this point of view appears fruitful. On the other hand, surely the choice of a functor in its equivalence class is not essential; and the emphasis on the inverse image vs. direct image functor is the matter of choice as well.

Given two abelian categories $\mathcal{A}$, $\mathcal{B}$, (equivalent to small categories) a **morphism** $f : \mathcal{B} \to \mathcal{A}$ is an equivalence class of right exact additive functors from $\mathcal{A}$ to $\mathcal{B}$. An **inverse image functor** $f^* : \mathcal{A} \to \mathcal{B}$ of f is simply a representative of f, which is usually assumed to be made.

An additive functor $f^* : \mathcal{B} \to \mathcal{A}$ between abelian categories is ([109])

• **continuous** if it has a right adjoint, say f_*;

• **flat** if it is continuous and exact;

• **almost affine** if it is continuous and its right adjoint f_* is faithful and exact;

• **affine** if it is almost affine, and its right adjoint f_* has its own right adjoint, say $f^!$, cf. **11.1**.

Morphism f is continuous (flat, almost affine, affine) if its inverse image functor f^* is. Some authors say that a functor F is continuous if it has a left adjoint instead, which means that they view F as a direct image functor f_* of some continuous morphism f. On the other hand, a continuous morphism is **coflat** if its direct image is exact, and **biflat** if it is flat and coflat. Usually one equips the categories with distinguished objects ("structure sheaves"); then the morphisms respect the distinguished object.

11.4a ROSENBERG ([109]) introduced an abstract notion of a **quasi-compact relative noncommutative scheme** $(\mathcal{A}, \mathcal{O})$ over a category $\mathcal{C}$ as an abelian category $\mathcal{A}$ with a distinguished object $\mathcal{O}$, finite biflat affine cover by localizations $Q_\lambda^* : \mathcal{A} \to \mathcal{B}_\Lambda$, with a continuous morphism g from $\mathcal{A}$ to $\mathcal{C}$ (think of it as a direct image of a morphism $X \to \operatorname{Spec} \mathbf{k}$) such that each $g_* \circ Q_{\lambda *} : \mathcal{B}_\lambda \to \mathcal{C}$ is affine. This guarantees application of many usual geometric procedure (for basic cohomological needs one does not need $f^!$). Such 'schemes' can be related to some spectra and some Grothendieck topologies on Aff(Ab) ([115]). Quotient spaces for comodule algebras over Hopf algebras may be sometimes constructed as nonaffine noncommutative schemes [120].

11.5 If R and S are rings and $_S B_R$ a $S - R$-bimodule, then the functor $f^B : M \mapsto {}_S B_R \otimes_R M$ is a right exact functor from $R - \operatorname{Mod}$ to $S - \operatorname{Mod}$. If $S = \mathbb{Z}$ then B_R is called flat right R-module if $f^B : M \mapsto B_R \otimes_R M$ is exact.

Proposition. (WATTS [143], EILENBERG [36]) *(i) Let R be a (not necessarily unital) ring, and f^* a flat endofunctor in the category of nonunital left R-modules. Then f^* is equivalent to the functor*

$$M \mapsto f^*(R_1) \otimes_R M,$$

where R_1 is the corresponding unital ring. In particular, $f^*(R_1)$ is flat as a left R-module.

(ii) Let R be a unital ring, and f^* a flat endofunctor in the category $R - \operatorname{Mod}$ of unital left R-modules. Then f^* is equivalent to the functor $M \mapsto f^*(R) \otimes_R M$.

(iii) Let $f^* : R - \operatorname{Mod} \to S - \operatorname{Mod}$ be a continuous functor. Then there is a $S - R$-bimodule $_S B_R$ such that f^* is equivalent to the functor $M \mapsto B \otimes_R M$.

Notice that when applied to localizations word "flat" is here used in the sense that Q^* is flat (weaker), and in some other works means that

$Q := Q_* Q^*$ is flat. The latter case, for $R - \mathrm{Mod}$, is the situation of Watts theorem, and it is known under the name of **perfect localizations**. Equivalently ([103]), the canonical forgetful functor from the localized category $(R - \mathrm{Mod})/\mathrm{Ker}Q$ to the modules over the localized ring $(QR) - \mathrm{Mod}$ is an equivalence of categories.

11.6 A **bicategory** (= weak 2-category) $\mathcal{A}$ consists of

(1) a class $\mathrm{Ob}\,\mathcal{A}$ of *objects* ('1-cells');

(2) for each pair of objects A, B a small category $\mathcal{A}(A, B)$; The objects of $\mathcal{A}(A, B)$ are called *arrows* (or 'morphisms' or '1-cells'), and the morphisms in $\mathcal{A}(A, B)$ are called *2-cells*;

(3) for each triple of objects A, B, C a bifunctor ('composition map')

$$\Phi_{A,B,C} : \mathcal{A}(A, B) \times \mathcal{A}(B, C) \to \mathcal{A}(A, C),$$

(4) for each object $A \in \mathcal{A}$, an arrow $1_A \in \mathcal{A}(A, A)$ ('identity arrow'); together with natural equivalences

$$a_{ABCD} : \Phi_{ACD}(\Phi_{ABC} \times \mathrm{Id}) \Rightarrow \Phi_{ABD}(\mathrm{Id} \times \Phi_{BCD}),$$
$$\lambda_{ABC} : \mathrm{Id} \times \Phi_{ABC} \Rightarrow \Phi_{ABC}, \qquad \rho_{ABC} : \Phi_{ABC} \times \mathrm{Id} \Rightarrow \Phi_{ABC},$$

satisfying some natural conditions. If $a_{ABCD}, \rho_{ABC}, \lambda_{ABC}$ are all identities, then the bicategory is called (strict) 2-category.

We omit further details in the definition, just to sketch the most important example to us: the bicategory of rings and bimodules $\mathfrak{Bim}$. Beforehand, notice that from any monoidal category $\tilde{\mathcal{C}} = (\mathcal{C}, \otimes, 1_{\mathcal{C}}, a, l, r)$ we can tautologically form a bicategory $\Sigma\tilde{\mathcal{C}}$ with one object $0 := \mathcal{C}$, and with $\Sigma\tilde{\mathcal{C}}(0, 0) := \tilde{\mathcal{C}}$ and with the composition $M \circ_{\Sigma\tilde{\mathcal{C}}} N := M \otimes N$; one further defines $a(M, N, P) := a_{0,0,0}(M, N, P)$, $l_A := \lambda_{0,0}(A)$ and $r_A := \rho_{0,0}(A)$. The definitions may be reversed to form a monoidal category out of any bicategory with a distinguished object 0 (forgetting the other objects). Hence monoidal categories may be viewed as bicategories with a single object; the notion of being 'strict' in both senses agrees as well.

The objects of $\mathfrak{Bim}$ are unital rings. For any $R, S \in \mathrm{Ob}\,\mathfrak{Bim}$ let $\mathfrak{Bim}(R, S)$ be the category of R-S-bimodules and (R, S)-bilinear mappings between them. The composition

$$\mathfrak{Bim}(R, S) \times \mathfrak{Bim}(S, T) \to \mathfrak{Bim}(R, T)$$

is given by the tensor product bifunctor $(M, N) \mapsto {}_R M_S \otimes_S {}_S N_T$, and the rest of the data is obvious. These data indeed define a bicategory.

The invertible 1-cells of $\mathfrak{Bim}$ are called MORITA equivalences. It has been observed in various applications of noncommutative geometry, for instance in physics, that Morita invariance is a common feature of natural constructions.

The Eilenberg-Watts' theorem identifies bimodules with flat functors. As a pair of adjoint functors, they resemble geometric morphisms among topoi. M. VAN DEN BERGH ([135]) defines a (generalized) bimodule to be any pair of adjoint functors between Grothendieck categories. He also considers right exact functors as so-to-say weak (version of) bimodules. Some situations, for example the duality for coherent sheaves involve functors for which the right and left adjoints ([60]). coincide. They are known as Frobenius functors ([25]). In the spirit of Van den Bergh's terminology, PAPPACENA calls *Frobenius bimodules* those adjoint pairs $F \dashv G$ where F is simultaneously left and right adjoint of G. In abstract homotopy theoretic setting, the existence of two-sided adjoints is studied with appropriate (Bousfield-type) localization arguments ([90, 96]).

One of motivations for this ([96]) is to extend the GROTHENDIECK duality theory for coherent sheaves on varieties to $\mathcal{D}$-modules. This may be viewed as an example of noncommutative geometry. Namely, the role of the structure sheaf $\mathcal{O}$ is played by the sheaf $\mathcal{D}$ of regular differential operators which is a sheaf of noncommutative $\mathcal{O}$-algebras (cf. [148, 102] for the viewpoint of noncommutative geometry at $\mathcal{D}$-modules). In triangulated categories, the Serre-Grothendieck duality is axiomatized as an existence of so-called SERRE functor ([18, 19, 20, 100]), with applications at the borderline between the commutative and noncommutative geometry.

It is a remarkable observation ([73]), that the noncommutative geometry via operator algebras, could be also organized around similar bicategories. Roughly speaking, operator algebras (C^*-algebras; von Neumann algebras respectively) are 0-cells, appropriate bimodules as 1-cells (Hilbert bimodules; correspondences), and bimodule morphisms as 2-cells; while the monoidal products of 1-cells are specific tensor products which became prominent earlier in noncommutative geometry a la Connes, and related K-theories (Rieffel interior tensor product of Hilbert bimodules; Connes fusion product). Invertible 1-cells are called **Morita equivalences** in all these cases. There are also analogues concerning regular bibundles over Lie groupoids, and also analogues in symplectic and Poisson geometry. The latter may be viewed as a quasiclassical limit of noncommutative geometry. For Morita equivalence of Poisson manifolds and similar notion of symplectic dual pairs see [74, 117].

12   Ore localization in filtered rings

After prerequisites on filtrations, we prove some general lemmas on localizations in filtered rings, mainly easy generalizations of some statements quoted without proof in [81] and in the manuscript [82][3].

We focus on *'negative'* filtrations. The main application in mind is the noncommutative deformation of commutative objects. Such filtrations arise from expanding the algebra operations in power series in the deformation parameter $q - 1$ ([4, 67, 78]). There is a more refined technique in algebraic analysis, *algebraic microlocalization*, see [136, 138] and the references in [136]. *'Positive'* filtrations involve different techniques than ours. In the study of noncommutative projective algebraic geometry there is a (negative) filtration related to deformation, but also a positive grading corresponding to the study of projective schemes. The latter grading may be refined to $\mathbb{Z}^{\times n}$-grading or grading with respect to the weight lattice P, as in the study of the quantum flag varieties ([81, 82]). If the latter, positive, grading complicates the picture, one restricts attention to homogeneous Ore sets only.

12.1 Given a (typically abelian) category $\mathcal{C}$, e.g. the category of modules over a ring **k**, a $\mathbb{Z}$-**filtration** on an object M in $\mathcal{C}$ is a nested (unbounded) sequence of subobjects $F_*M = \{\cdots \subset F_{n-1}M \subset F_nM \subset F_{n+1}M \cdots \subset M\}$. A $\mathbb{Z}$-filtered object in $\mathcal{C}$ an object with a $\mathbb{Z}$-filtration on it. All filtrations in this article are assumed to be **exhaustive** i.e. the supremum subobject $\sup_{n \in \mathbb{Z}} F_nM$ exists and equals M (e.g. for modules $M = \cup_n F_nM$).

Let $M = \cup_{n \in \mathbb{Z}} F_nM$ be a filtered **k**-module. The **degree** $d(m)$ of an element $m \in M$ is the smallest integer n, if it exists (otherwise $-\infty$), such that $m \in F_nM$ and $m \notin F_{n-1}M$. Conversely, if $d : M \to \{-\infty\} \cup \mathbb{Z}$ is subadditive $d(a + b) \leq d(a) + d(b)$, and $d(0) = -\infty$, then d is the degree function of a unique exhaustive filtration on M, indeed the one where $a \in F_nM$ iff $d(a) \leq n$. An (exhaustive) filtration is **separated** if $\cap_n F_nM = 0$. Then $d(m)$ is finite for all $m \neq 0$. *This will be our assumption from now on.*

12.1a A $\mathbb{Z}$-**filtered k-algebra** is a **k**-algebra (E, μ) with a filtration F_*E on E as a **k**-module where the multiplication μ restricted to $F_nE \times F_mE$ takes values within $F_{n+m}E$, for all n, m. This obviously generalizes to algebras in any monoidal category $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, 1_{\mathcal{C}})$ (notice that the resulting notion is different than if we consider these algebras as auxiliary *objects* in an abstract category $\mathcal{C}'$ of algebras, when **12.1a** applies, rather than as *algebras* in a *monoidal* category $\tilde{\mathcal{C}}$). For **k** $= \mathbb{Z}$ we talk about $\mathbb{Z}$-**filtered rings**.

Given a filtered **k**-algebra (F_*E, μ) a $\mathbb{Z}$-**filtered** F_*E-**module** is a $\mathbb{Z}$-

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filtered **k**-module F_*M such that F_nE acting on F_mM takes values within $F_{n+m}M$ for all n and m. In particular, F_*E is a -filtered module over itself.

Given a filtered **k**-algebra $E = \cup_n F_n E$, an **associated graded algebra** is the $\mathbb{Z}$-graded **k**-module $\operatorname{gr} E = \oplus_n (\operatorname{gr} E)_n := \oplus_n F_n E / F_{n-1} E$ with the multiplication defined as follows. The **symbol map** $^-: E \to \operatorname{gr} E$, $e \mapsto \bar{e}$, maps e to the class $\bar{e}$ of e in $F_{d(e)}E / F_{d(e)-1}E$. An element $c \in \operatorname{gr} E$ is in the image of the symbol map iff c is homogeneous. For $c = \bar{e}, c' = \bar{e}'$, the formula $cc' := \bar{e}\bar{e}' := \overline{ee}'$ does not depend on the choice of e, e'. Therefore it defines a binary operation. The additive extension of this operation is the associative multiplication on $\operatorname{gr} E$.

It is always true $d(e+e') \leq \max\{d(e), d(e')\}$, with equality if $d(e) \neq d(e')$. On the other hand, if $d(e) = d(e')$ then $d(e + e')$ may be in general anything less or equal $d(e)$, as e and e' may cancel in several of the top degrees. Consequently the symbol map is *not* additive. However...

12.2 *...if* $\operatorname{gr} E$ *is a domain, then* $d(ab) = d(a)d(b)$, *hence the symbol map* $E \to \operatorname{gr} E$ *is multiplicative.*

12.3 For any subset $S \subset E$ not containing 0, we can always define a filtration on the set $S \times E$, by formula $d(s, e) := d(e) - d(s)$. If S is left Ore, the localized ring $S^{-1}E$ may be constructed as in **5.6**, as certain quotient of $S \times E$. Hence we have a filtration on $S^{-1}E$ as a set with degree function $d(s^{-1}e) = \inf_{s'^{-1}e'=s^{-1}e} d(s', e')$. Recall that $(s, e) \sim (s', e')$ means $\exists \tilde{s} \in S$, $\exists \tilde{e} \in E$, $\tilde{s}s = \tilde{e}s'$ and $\tilde{s}e = \tilde{e}e'$.

If the degree function is multiplicative, e.g. E is a domain, then

$$
\begin{aligned}
d(s, e) &= d(e) - d(s) \\
&= d(e) - d(\tilde{e}) - (d(e) - d(\tilde{s})) \\
&= d(\tilde{s}r) - d(\tilde{s}s) \\
&= d(\tilde{e}e') - d(\tilde{e}s') \\
&= d(e') - d(s') \\
&= d(s', e'),
\end{aligned}
$$

hence taking the infimum in the expression for $d(s^{-1}e)$ is superfluous, as all the representatives of $s^{-1}e$ give the same result. Therefore the degree is well-defined by $d(s^{-1}e) := d(e) - d(s)$.

The symbol image of a set $S \subset E$ is denoted by $\overline{S}$. If S is left Ore in E, and $\operatorname{gr} E$ is a domain, then $\overline{S}$ is clearly left Ore in $\operatorname{gr} E$.

12.4 Lemma. *If the symbol map* $E \to \operatorname{gr} E$ *is multiplicative, then the induced degree function on* $S^{-1}E$ *is multiplicative as well.*

12.5 Proposition. ([81], II 3.2)*(i) We have a well-defined map* θ : $\bar{S}^{-1}\operatorname{gr} E \to \operatorname{gr} S^{-1}E$ *given by* $(\bar{s})^{-1}\bar{e} \mapsto \overline{s^{-1}e}$.

(ii) This map is an isomorphism of graded rings.

Proof.   (i) Let $s_1, s_2 \in S$ with $\bar{s}_1 = \bar{s}_2$. By $s_1^{-1}s_2 = 1 + s_1^{-1}(s_2 - s_1)$ we get $s_1^{-1}e = s_2^{-1}e + s_1^{-1}(s_2 - s_1)s_2^{-1}e$. Using **12.4** and $d(s_2 - s_1) < d(s_1)$, we see that for each $0 \neq e \in E$, $s_1^{-1}(s_2 - s_1)s_2^{-1}e$ is lower degree than $s_2^{-1}e$. Thus $\overline{s_1^{-1}e} = \overline{s_2^{-1}e} \in \overline{S^{-1}E}$. In the same vein, but easier, we see that $\overline{s^{-1}e}$ does not depend on the choice of $e \in \bar{e}$.

Finally, choose different classes $\bar{t}$ and $\bar{f}$ with $(\bar{t})^{-1}\bar{f} = (\bar{s})^{-1}\bar{e}$. That is $\exists \bar{s}_* \in \bar{S}, \exists \bar{e}_* \in \operatorname{gr} E$ with $\bar{s}_* \bar{t} = \bar{e}_* \bar{s} \in \bar{S}$ and $\bar{s}_* \bar{f} = \bar{e}_* \bar{e}$. Then $\overline{s_* t} = \overline{e_* s}$ and $\overline{s_* f} = \overline{e_* e}$ for any choice of representatives e_*, s_* of $\bar{e}_*, \bar{s}_*$. Hence there are r_1, r_2 of lower degrees than $e_* s, e_* e$ respectively, such that $s_* t = e_* s + r_1$ and $s_* f = e_* e + r_2$. Then $t^{-1}f = (s_* t)^{-1}s_* f = (e_* s + r_1)^{-1}(e_* e + r_2)$ which by the above equals $(e_* s)^{-1}(e_* e) = s^{-1}e$ up to elements of lower order, *provided that $e_* s \in S$*. As $e_* s + r_1 \in S$, this is always true if S is *saturated*, see below. However, the conclusion follows without that assumption. Indeed, by the left Ore condition, choose $s^\sharp, e^\sharp$ with $S \ni s^\sharp(e_* s + r_1) = e^\sharp s$. Then $e^\sharp e = e^\sharp s s^{-1}e = s^\sharp e_* e + s^\sharp r_1 s^{-1}e$, where, by the multiplicativity, $s^\sharp r_1 s^{-1}e$ is of lower order. Consequently, $t^{-1}f = [s^\sharp(e_* s + r_1)]^{-1}s^\sharp(e_* e + r_2) = (e^\sharp s)^{-1}e^\sharp e +$ lower order $= s^{-1}e +$ lower order, as required.

(ii) Since both the degree of $(\bar{s})^{-1}\bar{e}$ and of $\overline{s^{-1}e}$ are $d(e) - d(s)$, this map respects the grading. The obvious candidate $\overline{s^{-1}e} \mapsto (\bar{s})^{-1}\bar{e}$ for the inverse is well-defined by more straightforward reasons than the map θ. Namely, if $\overline{t^{-1}f} = \overline{s^{-1}e}$ then $\exists h$ of lower order with $t^{-1}f = s^{-1}e + h = s^{-1}(e + sh)$. As $\bar{e} = \overline{e + sh}$ it is enough to check the case $h = 0$. For some $s_* \in S$, $r_* \in r$ we have $s_* t = r_* s \in S$ and $s_* f = r_* e$. Then $\bar{s}_* \bar{t} = \bar{r}_* \bar{s} \in \bar{S}$ and $\bar{s}_* \bar{f} = \bar{r}_* \bar{e}$, hence $\bar{t}^{-1}\bar{f} = \bar{s}^{-1}\bar{e}$, as required.

12.6 Let $N = \cup_{k \in \mathbb{Z}}F_k N$, be a right and $M = \cup_{k \in \mathbb{Z}}F_k M$ a left filtered E-module, then $N \otimes_{\mathbf{k}} M$ is filtered with respect to the unique degree function additively extending formulas $d(n \otimes_k m) = d(n) + d(m)$. The canonical quotient map $p_E : N \otimes_{\mathbf{k}} M \to N \otimes_E M$ induces the filtration $F_k(N \otimes_E M) := p_E(F_k(N \otimes_{\mathbf{k}} M))$. If N is a filtered $E' - E$-bimodule, one obtains a filtration of $N \otimes_E M$ as a left E'-module. In particular, *given $E = \cup_k F_k E$, where $\operatorname{gr} E$ is a domain, and given a filtered left E-module $M = \cup_k F_k M$, any Ore localization $S^{-1}M = S^{-1}E \otimes_E M$ is a filtered left $S^{-1}E$-module with the degree function*

$$d(s^{-1}m) = d(s^{-1} \otimes_E m) = d(s^{-1}1_E) + d(m) = d(m) - d(s).$$

12.7 Lemma.   *If the symbol map $E \to \operatorname{gr} E$ is multiplicative, and M a filtered left E-module, then the degree functions are compatible with action in the sense that $d_M(e.m) = d_E(e)d_M(m)$. Furthermore, for any left Ore*

set $S \subset E$,

$$
\begin{aligned}
d_{S^{-1}M}(s^{-1}e.t^{-1}m) &= d_{S^{-1}E}(s^{-1}e)d_{S^{-1}M}(t^{-1}m) \\
&= d_E(e) + d_M(m) - d_E(s) - d_E(t).
\end{aligned}
$$

12.8 Proposition. *(i) For a filtered ring E, for which $\mathrm{gr}\, E$ is a domain, and any filtered E-module M, we have a well-defined map $\theta_M : \bar{S}^{-1}\mathrm{gr}\, M \to \mathrm{gr}\, S^{-1}M$ given by $(\bar{s})^{-1}\bar{m} \mapsto \overline{s^{-1}m}$.*

(ii) θ_M is an isomorphism of graded $\mathrm{gr}\, S^{-1}E = \overline{S}^{-1}\mathrm{gr}\, E$-modules.

The proof is by the same techniques as **12.5**. The compatibility with the action **12.7** replaces the multiplicativity, and the formula (9) for the equivalence relation $\sim$ on $S \times M$ (with $(S \times M/ \sim) \cong S^{-1}M$) replaces the equivalence relation $\sim$ from **4.13** on $S \times R$ in that proof.

12.9 Ore conditions recursively. *(i) Let S be a multiplicative set in a ring E with an exhaustive filtration*

$$
F_* E = \{\ldots \subset F_{-r}E \subset F_{-r+1}E \subset \ldots \subset F_{-1}E \subset F_0 E \subset F_1 E \subset \cdots \subset E\}.
$$

Let S satisfy the **bounded below filtered-relative left Ore condition** *in $F_* E$:*

$\exists r, \infty > r \geq -n, \forall s \in S, \forall k, -r \leq k \leq n, \forall e \in F_k E, \exists s' \in S, \exists e' \in E$ such that $s'e - e's \in F_{k-1}E$ if $k > -r$, and $s'e - e's = 0$ if $k = -r$.

Then S satisfies the left Ore condition for S in E.

(ii) Assume that S is **bounded filtered left reversible** *in $F_* E$:*

$\exists r < \infty, \forall e_k \in F_k E$, if $\exists s \in S$ with $e_k s \in F_{k-1}E$ then $\exists s' \in S$ such that $s'e_k \in F_{k-1}E$ if $k > -r$, and $s'e_k = 0$ if $k = -r$.

Then S is left reversible in E.

Proof. (i) Let $s \in S$ and $e = e_n \in F_n E$. By induction, we can complete sequences $e_n, \ldots, e_{-r}, e'_n, \ldots, e'_{-r}$ (here $e_k, e'_k \in F_k E$) and $s'_n, \ldots, s'_{-r} \in S$, with $e'_k s = s'_k e_k - e_{k-1}$ for all k with $e_{-r-1} := 0$. By descending induction on k,

$$
(e'_k + s'_k e'_{k+1} + \ldots + s'_k \cdots s'_{n-1}e'_n)\, s = s'_k \cdots s'_n e_n - e_{k-1},
$$

for each $k > -r$, and finally,

$$
(e'_{-r} + s'_{-r}e'_{-r+1} + \ldots + s'_{-r}s'_{-r+1} \cdots s'_{n-1}e'_n)\, s = s'_{-r} \cdots s'_n e_n.
$$

(ii) Suppose $e \in F_k E = E$ and $es = 0$ for some $s \in S$. It is sufficient to inductively choose a descending sequence and $s'_{k+1} = 1, s'_k, s'_{k-1}, \ldots, s'_{-r} \in S$, with requirements $s'_j e \in F_{j-1}E$ for all $j > -r$ and $s'_{-r}e = 0$. Suppose we

have chosen $s_k, \ldots, s_{j+1}$. Then $(s'_{j+1}e)s = s'_{j+1}(es) = 0$ with $s'_{j+1}e \in F_j E$, hence by the assumption there exist some $\tilde{s} \in S$ such that $\tilde{s}s'_{j+1}e \in F_{j-1}E$. Set therefore $s'_j := \tilde{s}s'_{j+1} \in S$.

12.9a Let F_*E be an exhaustive filtration of E with $F_{-r}E = 0$ for some finite r, and $S \subset E$ be a multiplicative set. If its image $\bar{S}$ under the symbol map satisfies the left Ore condition in $\operatorname{gr} E$, then the conditions in **12.9** hold. Hence S satisfies the left Ore condition in E as well.

12.9b Let $t \in E$ be a regular element $(tE = Et)$ in a ring E. Then for each $n > 0$ the ideal $t^n E$ is 2-sided, hence $E_n := E/(t^n E)$ is a quotient ring in which the element t is nilpotent of order less or equal n. Rule $F_{-k}E_n = (t^k E)/(t^n E) \subset E/(t^n E) \equiv E_n$ defines a bounded 'negative' filtration

$$F_*E_n = \{0 = F_{-n}E_n \subset \ldots \subset F_{-1}E_n \subset F_0 E_n = E_n\}$$

in which (the image of) t is of degree -1. If $\operatorname{gr} E$ is a domain then *both* the symbol map $E \to \operatorname{gr} E$, and its truncation $E_n \to \operatorname{gr} E_n$ are multiplicative.

12.10 Theorem. *Let S be a left Ore set in some ring $E_n = F_0 E_n$ with a bounded negative filtration $F_\bullet E_n$. Suppose $S' \subset E_n$ is a multiplicative set such that $s' \in S' \cap F_j E$ iff $\exists b \in F_{j-1}E$ such that $s' = s - b$. Then S' is left Ore as well and $S^{-1}E_n = (S')^{-1}E_n$ as graded rings.*

Proof. Since the left Ore localization is a universal object in the category $\mathcal{C}_l(E_n, S)$ (cf. Chap.4) it is enough to see that a map of rings $j : E_n \to Y$ is in it iff it is in $\mathcal{C}_l(E_n, S')$. If $j(s)$ is invertible in Y, let $c = j(s)^{-1}j(b)$. Mapping j induces a (non-separated in general) filtration on Y such that j is a map of filtered rings, by taking the degree to be the infimum of expressions $d(e) - d(t)$ for elements which can be represented in the form $j(t)^{-1}j(e)$ and $-\infty$ otherwise. With our numerical constraints on the degree, for nonvanishing $e \in E_n$ this difference can not be less than $-n$. As $d(c) < 0$ we obtain $d(c^n) < n - 1$, hence $c^n = 0$. Thus we can invert $j(s)^{-1}(j(s - b)) = 1 - c$ to obtain the geometrical progression $\sum_{j=0}^{n-1} c^j$. Then $\sum_{j=0}^{n-1} c^j j(s)^{-1}j(s-b) = 1$ hence $j(s - b)$ is invertible in Y.

It remains to check that $se = 0$ for some $s \in S$ iff $\exists s'' \in S'$ with $s''e = 0$. We proceed by induction on the degree j of e starting at $-n$ where $s'e - se \in F_{-n-1} = 0$ for $s' = s - b$ with the degree of b smaller than of s' hence negative. For any j, $s'e = (s - b)e = -be$ has the degree at most $j - 1$. On the other hand, by the left Ore condition, we can find $s_* \in S$, and $b_* \in E$ with $s_* be = b_* se = 0$, hence $s_*(s'e) = 0$. Set $e' := s'e$. Since $s_* e' = 0$ with $d(e') < d(e)$, by the inductive assumption there exists $s'_* \in S'$ with $s'_* s'e = s'_* e' = 0$. Set $s'' := s'_* s'$.

The induced grading on the two localized rings is the same after the identification, because the symbol maps evaluate to the same element on s

and $s' = s + b$ (or, alternatively, after the identification, the gradings on the localization are induced by the *same* ring map).

Definition. *A multiplicative subset $S \subset E_n$ is*
- **admissible** *if $\forall s \in S$, $0 \neq \bar{s} \in \operatorname{gr} E_n$;*
- **saturated** *if $S = \{s \in E_n \,|\, \bar{s} \in \bar{S}\}$.*

12.11 Corollary. *Let $E, t, E_n, F_\bullet E_n$ be as in **12.9**. Suppose $\operatorname{gr} E$ is a commutative domain. Let S be a multiplicative subset in E_n. Then*

a) S is left and right Ore.

b) $S^{-1}E_n \neq 0$ iff S is admissible.

c) $S^{-1}E_n$ depends only on $\bar{S} \subset \bar{E}$.

d) $S^{-1}E_n$ is filtered by powers of t and $(S^{-1}E_n)/\langle t \rangle \cong \bar{S}^{-1}E_n$.

e) Any two saturated Ore sets S, T are compatible, i.e. $S^{-1}T^{-1}E_n \cong T^{-1}S^{-1}E_n$ and $ST = \{st \,|\, s \in S, t \in T\}$ is also saturated.

f) Let S be admissible. Then $\operatorname{gr}(S^{-1}E_n) \cong \bar{S}^{-1}\operatorname{gr}E_n$. In particular, $\overline{S^{-1}E_n} \cong \overline{S}^{-1}\overline{E}_n$.

Sketch of the proof. a) follows as a simple case of **12.9**; b) is trivial; c) follows by **12.10**; d) is evident; f) follows from **12.8** after truncating both sides from E to the quotient filtered ring E_n (it is not a special case of **12.8**, though, as E_n is *not* a domain); e) Because T is saturated, $(T^{-1}E)_n \cong T^{-1}E_n$. In $T^{-1}E_n$, set S is still multiplicative, hence by a) applied to $(T^{-1}E)_n$ it is left Ore. This is equivalent to compatibility (cf. Sec. 10).

13   Differential Ore condition

An extensive literature is dedicated to differential structures of various kind associated to objects of noncommutative geometry: derivations and rings of regular differential operators on NC rings, 1st and higher order differential calculi, with and without (bi)covariance conditions, NC connections and de Rham complexes etc.

13.1   *Let $\partial : R \to R$ be an R-valued derivation on R and S a left Ore set in R. Then the formula*

$$\bar{\partial}(s^{-1}r) = s^{-1}\partial(r) - s^{-1}\partial(s)s^{-1}r, \quad s \in S, r \in R, \tag{15}$$

defines a derivation $\bar{\partial} : S^{-1}R \to S^{-1}R$.

The same conclusion if we started with $\partial : R \to S^{-1}R$ instead.

Proof. 1. $\bar{\partial}$ IS WELL DEFINED.
Suppose $s^{-1}r = t^{-1}r'$ for some $r, r' \in R$, $s, t \in S$. Then

$$\exists \tilde{s} \in S, \exists \tilde{r} \in R, \quad \tilde{s}t = \tilde{r}s, \quad \tilde{s}r' = \tilde{r}r.$$

$$s^{-1} = t^{-1}\tilde{s}^{-1}\tilde{r}$$

$$
\begin{aligned}
t^{-1}\partial(r') &= t^{-1}\tilde{s}^{-1}\tilde{s}\,\partial(r') \\
&= t^{-1}\tilde{s}^{-1}[\partial(\tilde{s}r') - \partial(\tilde{s})r'] \\
&= t^{-1}\tilde{s}^{-1}[\partial(\tilde{r}r) - \partial(\tilde{s})r'] \\
t^{-1}\partial(t) &= t^{-1}\tilde{s}^{-1}\tilde{s}\,\partial(t) \\
&= t^{-1}\tilde{s}^{-1}[\partial(\tilde{s}t) - \partial(\tilde{s})t] \\
&= t^{-1}\tilde{s}^{-1}[\partial(\tilde{r}s) - \partial(\tilde{s})t]
\end{aligned}
$$

$$
\begin{aligned}
\bar{\partial}(t^{-1}r') &= t^{-1}\partial(r') - t^{-1}\partial(t)t^{-1}r' \\
&= t^{-1}\tilde{s}^{-1}[\partial(\tilde{r}r) - \partial(\tilde{s})r'] - t^{-1}\tilde{s}^{-1}[\partial(\tilde{r}s) - \partial(\tilde{s})t]t^{-1}r' \\
&= t^{-1}\tilde{s}^{-1}\partial(\tilde{r}r) - t^{-1}\tilde{s}^{-1}\partial(\tilde{r}s)t^{-1}r' \\
&= t^{-1}\tilde{s}^{-1}\partial(\tilde{r}r) - t^{-1}\tilde{s}^{-1}\partial(\tilde{r}s)s^{-1}r \\
&= t^{-1}\tilde{s}^{-1}\partial(\tilde{r})r + t^{-1}\tilde{s}^{-1}\tilde{r}\partial(r) \\
&\qquad -t^{-1}\tilde{s}^{-1}\partial(\tilde{r})ss^{-1}r - t^{-1}\tilde{s}^{-1}\tilde{r}\partial(s)s^{-1}r \\
&= s^{-1}\partial(r) - s^{-1}\partial(s)s^{-1}r \\
&= \bar{\partial}(s^{-1}r)
\end{aligned}
$$

2. $\bar{\partial}$ IS A DERIVATION. We have to prove that for all $s,t \in S$ and $r,r' \in R$

$$\bar{\partial}(s^{-1}rt^{-1}r') = \bar{\partial}(s^{-1}r)\,t^{-1}r' + s^{-1}r\bar{\partial}(t^{-1}r'). \tag{16}$$

The argument of $\bar{\partial}$ on the left hand side has to be first changed into a left fraction form before we can apply the definition of $\bar{\partial}$. By the left Ore condition, we can find $r_* \in R$, $s_* \in S$ such that $r_*t = s_*r$ i.e. $rt^{-1} = s_*^{-1}r_*$.

We first prove identity (16) in the case $s = r' = 1$ i.e.

$$\bar{\partial}(rt^{-1}) = \partial(r)\,t^{-1} + r\bar{\partial}(t^{-1}). \tag{17}$$

The left-hand side of (17) is

$$
\begin{aligned}
\bar{\partial}(rt^{-1}) &= \bar{\partial}(s_*^{-1}r_*) \\
&= s_*^{-1}\partial(r_*) - s_*^{-1}\partial(s_*)s_*^{-1}r_* \\
&= s_*^{-1}\partial(r_*) + \bar{\partial}(s_*^{-1})r_*.
\end{aligned}
$$

The right-hand side of (17) is

$$
\begin{aligned}
\partial(r)t^{-1} - rt^{-1}\partial(t)t^{-1} &= \partial(r)t^{-1} - s_*^{-1}r_*\partial(t)t^{-1} \\
&= \partial(r)t^{-1} - s_*^{-1}\partial(r_*t)t^{-1} - s_*^{-1}\partial(r_*)tt^{-1} \\
&= \partial(r)t^{-1} - \bar{\partial}(s_*^{-1}r_*t)t^{-1} + \bar{\partial}(s_*^{-1})r_* - s_*^{-1}\partial(r_*) \\
&= \partial(r)t^{-1} - \partial(r)t^{-1} - \bar{\partial}(s_*^{-1})r_* - s_*^{-1}\partial(r_*) \\
&= \bar{\partial}(s_*^{-1})r_* - s_*^{-1}\partial(r_*).
\end{aligned}
$$

Hence (17) follows. Using (17), we prove (16) directly:

$$
\begin{aligned}
\bar{\partial}(s^{-1}rt^{-1}r') &= \bar{\partial}((s_*s)^{-1}r_*r') \\
&= (s_*s)^{-1}\partial(r_*r') - (s_*s)^{-1}\partial(s_*s)(s_*s)^{-1}r_*r' \\
&= s^{-1}s_*^{-1}\partial(r_*)r' + s^{-1}s_*^{-1}r_*\partial(r') \\
&\quad -s^{-1}s_*^{-1}\partial(s_*)s_*^{-1}r_*r' - s^{-1}\partial(s)s^{-1}s_*^{-1}r_*r' \\
&= s^{-1}s_*^{-1}\partial(r_*)r' + s^{-1}t^{-1}r\partial(r') + s^{-1}\bar{\partial}(s_*^{-1})r_*r' + \bar{\partial}(s^{-1})s_*^{-1}r_*r' \\
&= s^{-1}\bar{\partial}(s_*^{-1}r_*)r' - s^{-1}\bar{\partial}(s_*^{-1})r_*r + s^{-1}rt^{-1}\partial(r') + \bar{\partial}(s^{-1})rt^{-1}r' \\
&= s^{-1}\bar{\partial}(rt^{-1}) - s^{-1}\bar{\partial}(s_*^{-1})r_*r + s^{-1}rt^{-1}\partial(r') + \bar{\partial}(s^{-1})rt^{-1}r' \\
&\stackrel{(17)}{=} s^{-1}\partial(r)t^{-1}r' + s^{-1}r\bar{\partial}(t^{-1})r' + s^{-1}rt^{-1}\partial(r') + \bar{\partial}(s^{-1})rt^{-1}r' \\
&= \bar{\partial}(s^{-1}r)t^{-1}r' + s^{-1}r\bar{\partial}(t^{-1}r').
\end{aligned}
$$

Standard textbooks have incomplete proofs of **13.1**, e.g. [32, 116].

13.2 Definition. *A **Poisson bracket** on a unital associative **k**-algebra is an antisymmetric bilinear operation* $\{,\} : A\otimes A \to A$ *satisfying the Jacobi identity* $\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\} = 0$ *for all* $f,g,h \in A$ *and such that for each* f, *$\mathbf{k}$-linear map* $X_f : g \mapsto \{f,g\}$ *is a $\mathbf{k}$-derivation of A. A **Poisson algebra** is a commutative algebra with a Poisson bracket.*

Proposition. *Let A be a $\mathbf{k}$-algebra with a Poisson bracket $\{,\}$, and $S \subset A\backslash\{0\}$ a central multiplicative set. Then*

(i) $S^{-1}A$ posses a bilinear bracket $\{,\} = \{,\}^S$ such that the localization map $\iota_S : A \to S^{-1}A$ intertwines the brackets: $\{,\}^S \circ (\iota_S \otimes_{\mathbf{k}} \iota_S) = \iota_S \circ \{,\}$.

(ii) If either $\{s,t\} \in \operatorname{Ker}\iota_S$ for all $s,t \in S$, or if A is commutative, then there is a unique such bracket $\{,\}^S$ which is, in addition, skew-symmetric.

(iii) If A is commutative then this unique $\{,\}^S$ is a Poisson bracket.

Proof. (i) Each X_b by **13.1** induces a unique derivation $X_b^S = \bar{\partial}$ on $S^{-1}A$ by (15) for $\partial = X_b$. Map $b \mapsto X_b^S$ is $\mathbf{k}$-linear by uniqueness as $X_b^S + X_c^S$ is a derivation extending X_{b+c} as well. For each $s^{-1}a \in S^{-1}A$ define $\mathbf{k}$-linear map $Y_{s^{-1}a} : A \to S^{-1}A$ by

$$
Y_{s^{-1}a} : b \mapsto -X_b^S(s^{-1}a) = -s^{-1}\{b,a\} + s^{-1}\{b,s\}s^{-1}a.
$$

Because s is central, $Y_{s^{-1}a}$ is a $\mathbf{k}$-linear derivation. Namely,

$$
\begin{aligned}
Y_{s^{-1}a}(bc) &= -s^{-1}\{bc,a\} + s^{-1}\{bc,s\}s^{-1} \\
&= -s^{-1}\{b,a\}c - s^{-1}b\{c,a\}+ \\
&\quad + s^{-1}b\{c,s\}s^{-1}a + s^{-1}\{b,s\}cs^{-1}a,
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
Y_{s^{-1}a}(b)c + bY_{s^{-1}a}(c) &= -s^{-1}\{b,a\}c + s^{-1}\{b,s\}s^{-1}ac- \\
&\quad - bs^{-1}\{c,a\} + bs^{-1}\{c,s\}s^{-1}a.
\end{aligned}
$$

Hence $Y_{s^{-1}a}$ extends to a derivation $Y^S_{s^{-1}a}$ on $S^{-1}A \to S^{-1}A$ by formula (15) as well. Define $\{s^{-1}a, t^{-1}b\} := Y^S_{s^{-1}a}(t^{-1}b)$.

(ii) To show the skew-symmetry, we calculate,

$$
\begin{aligned}
Y^S_{s^{-1}a}(t^{-1}b) &= t^{-1}Y_{s^{-1}a}(b) - t^{-1}Y_{s^{-1}a}t^{-1}b \\
&= -t^{-1}X^S_b(s^{-1}a) + t^{-1}X^S_t(s^{-1}a)t^{-1}b \\
&= -t^{-1}s^{-1}X_b(a) + t^{-1}s^{-1}X_b(s)s^{-1}a \\
&\quad + t^{-1}s^{-1}X_t(a)t^{-1}b - t^{-1}s^{-1}X_t(s)s^{-1}at^{-1}b.
\end{aligned}
$$

$$
\begin{aligned}
Y^S_{t^{-1}b}(s^{-1}a) &= s^{-1}Y_{t^{-1}b}(a) - s^{-1}Y_{t^{-1}b}(s)s^{-1}a \\
&= -s^{-1}X^S_a(t^{-1}b) + s^{-1}X^S_s(t^{-1}b)s^{-1}a \\
&= -s^{-1}t^{-1}X_a(b) + s^{-1}t^{-1}X_a(t)t^{-1}b \\
&\quad + s^{-1}t^{-1}X_s(b)s^{-1}a - s^{-1}t^{-1}X_s(t)t^{-1}bs^{-1}a.
\end{aligned}
$$

Using $X_b(a) = -X_a(b)$ etc. and centrality of s, t we see that the first 3 terms in $Y^S_{s^{-1}a}(t^{-1}b)$ match with negative sign the first 3 terms (in order 1,3,2) in expression for $Y^S_{t^{-1}b}(s^{-1}a)$. If a and b mutually commute, the 4th term agrees the same way, and if they don't but $X_s(t) = 0$ in the localization $S^{-1}A$, then they are simply 0, implying skew-symmetry $\{s^{-1}a, t^{-1}b\} + \{t^{-1}b, s^{-1}a\} = 0$.

Uniqueness: $Z_{s^{-1}a}(t^{-1}b) := \{s^{-1}a, t^{-1}b\}$ defines a derivation $Z_{s^{-1}a}$ on $S^{-1}A$, which restricts to a derivation $Z_{s^{-1}a}| : A \to S^{-1}A$. On the other hand, $s^{-1}a \mapsto Z_{s^{-1}a}(b)$ is $-X^S_b$ by its definition. Hence the value of $Z_{s^{-1}a}|$ is determined at every b, and by **13.1** this fixes $Z_{s^{-1}a}$.

(iii) We'll prove that if the Jacobi rule holds for given (a, b, c) and (s, b, c) in $S^{-1}A^{\times 3}$, then it follows for $(s^{-1}a, b, c)$ provided s is invertible. By symmetry of the Jacobi rule and by renaming $s^{-1}a \mapsto a$ we infer that it follows for $(s^{-1}a, t^{-1}b, c)$, as well, and finally for the general case by one more application of this reasoning. Thus we only need to show that $\mathrm{Jacobi}(a, b, c)$ implies $\mathrm{Jacobi}(s^{-1}a, b, c)$. For commutative $S^{-1}A$ this is a straightforward calculation, using the Jacobi identity, lemma above and skew-symmetry. We name the summands:

$$
\begin{aligned}
\{s^{-1}a, \{b, c\}\} &= s^{-1}\{a, \{b, c\}\} - s^{-2}\{s, \{b, c\}\}a =: (A1) + (A2) \\
\{b, \{c, s^{-1}a\}\} &= s^{-1}\{b, \{c, a\}\} - s^{-2}\{b, s\}\{c, a\} - s^{-2}\{b, \{c, s\}\}a \\
&\quad - s^{-2}\{c, s\}\{b, s\}a - s^{-1}\{c, s\}\{b, a\} \\
&=: (B1) + (B2) + (B3) + (B4) + (B5) \\
\{c, \{s^{-1}a, b\}\} &= s^{-1}\{c, \{a, b\}\} - s^{-2}\{c, s\}\{a, b\} - s^{-2}\{c, \{s, b\}\}a \\
&\quad + s^{-3}\{s, b\}\{c, s\}a - s^{-2}\{s, b\}\{c, a\} \\
&=: (C1) + (C2) + (C3) + (C4) + (C5).
\end{aligned}
$$

Then $(A1) + (B1) + (C1) = 0$ and $(A2) + (B3) + (C3) = 0$ by Jacobi for (a, b, c), and (b, c, s) respectively. By skew-symmetry $(B2) + (C5) = 0$, $(B5) + (C2) = 0$ and $(B4) + (C4) = 0$ which finishes the proof.

This fact for A (super)commutative is used for example in the theory of integrable systems, sometimes in connection to 'quantization' which is a rich source of examples in noncommutative geometry.

13.3 Let $(R, \cdot, +)$ be a ring (**k**-algebra), not necessarily unital. A *first order differential calculus* (FODC) is a $R-R$-bimodule $\Omega^1(R)$ together with an additive (**k**-linear) map $d : R \to \Omega^1(R)$ satisfying Leibnitz identity

$$d(ab) = d(a)b + ad(b), \quad a, b \in R,$$

and such that $\Omega^1(R)$ is generated by differentials dr, $r \in R$ as a *left module*. Define a category $\mathfrak{Fodc}$: objects are pairs of a ring R and a FODC $(\Omega^1(R), d)$ on R. A morphisms is a pair $(f, e) : (R, \Omega^1(R), d) \to (R', \Omega^1(R'), d')$ of a ring map $f : R \to R'$ and a map $e : \Omega^1(R) \to \Omega^1(R')$ of $R - R$-bimodules such that $e \circ d = d' \circ f$. Fixing R and allowing only morphisms of the form (Id_R, d) we obtain a (non-full) subcategory $\mathfrak{Fodc}_R$ of $\mathfrak{Fodc}$. If R is unital, then $(\mathrm{Ker}(R \otimes_\mathbf{k} R \xrightarrow{\cdot} R, d)$ where $da = 1 \otimes a - a \otimes 1$, and the R-bimodule structure is ${}_R R \otimes_\mathbf{k} R_R$, is an initial object of that category.

Two objects $c_R = (R, \Omega^1(R), d)$, $c_{R'} = (R', \Omega^1(R'), d')$ in $\mathfrak{Fodc}$ are **compatible along** $f : R \to R'$ if there is an e such that $(f, e) \in \mathfrak{Fodc}(c_R, c_{R'})$.

Differential calculi restrict: Given $c_{R'} \in \mathfrak{Fodc}_{R'}$ and f as above, define $f^1_\Omega \Omega^1(R')$ to be the smallest additive subgroup of $\Omega^1(R')$ containing all the elements of the form $f(a)\partial'(f(b))$, $a, b \in R$. It appears to be an $R - R$-bimodule. Define $f^\sharp(c_{R'}) := (R, f^1_\Omega \Omega^1(R'), d' \circ f)$. Then $f^\sharp(c_{R'}) \in \mathfrak{Fodc}_R$ because $\partial(b).c = \partial'(f(b))f(c) = \partial'(f(bc)) - f(b)\partial'(f(c)) = \partial(bc) - b.\partial(c) \in f^\sharp \Omega^1(R')$. where $\partial = \partial' \circ f : R \to f^\sharp \Omega^1(R')$ is the restricted differential. Note the decomposition of $(f, e) : c_R \to c_{R'}$ into $(f, e) : c_R \to f^\sharp c_{R'}$ and $(\mathrm{id}_{R'}, \mathrm{incl}) : f^\sharp c_{R'} \to c_{R'}$, where $\mathrm{incl} : f^1_\Omega \Omega^1(R') \to \Omega^1(R')$ is the inclusion of R'-bimodules.

Unlike restricting, there is *no general recipe for extending* the calculus along ring maps $f : R \to R'$, except for the special case when $R' = S^{-1}R$ and $\Omega^1 R = {}_R R_R$, treated in **13.1**. That case is of central importance in study of the regular differential operators and D-modules over noncommutative spaces ([79, 80, 81]). We'll just mention a slight generalization.

13.4 Theorem. *Let $S \subset R$ be a left Ore set in a ring R, and suppose $\{x \in \Omega^1(R) \,|\, \exists t \in S, xt = 0\} = 0$.*

The following are then equivalent:

(i) The $S^{-1}R$-R-bimodule structure on $S^{-1}\Omega^1(R) \equiv S^{-1}R \otimes_R \Omega^1(R)$ extends to an (actually unique) $S^{-1}R$-bimodule structure which may carry a differential $d_S : S^{-1}R \to S^{-1}\Omega^1(R)$ such that the pair of localization maps

$(\iota_S, \iota_{S,\Omega^1(R)})$ *is a morphism in* $\mathfrak{Fodc}$ *(i.e. 'the calculi are compatible along the localization').*

(ii) The **differential Ore condition** *is satisfied:*

$$\boxed{\forall t \in S, \ \forall r \in R, \ \exists s \in S, \ \exists \omega \in \Omega^1(R), \quad s \, dr = \omega t.}$$

Proof. (i) $\Rightarrow$ (ii). If $S^{-1}\Omega^1(R) \equiv S^{-1}R \otimes_R \Omega^1(R)$ is a $S^{-1}R$-bimodule then $(dr)t^{-1} \in S^{-1}\Omega^1(R)$ for $t \in S$, $r \in R$. All the elements in $S^{-1}\Omega^1(R)$ are of the form $s^{-1}\omega$ where $s \in S$ and $\Omega \in \Omega^1(R)$. Hence $\exists s \in S$, $\exists \omega \in \Omega^1(R)$ such that $s \, dr = \omega t$ in the localization. By **7.4** this means $s \, dr = \omega t + \omega'$ in $\Omega^1(R)$, where $s'\omega' = 0$ for some $s' \in S$. Pre-multiplying by s' we obtain $(s's)dr = (s'\omega)t$, with required form.

(ii) $\Rightarrow$ (i). The right $S^{-1}R$ action if it exists is clearly forced by

$$s_1^{-1}ad(r)t^{-1}b = s_1^{-1}as^{-1}\omega b \tag{18}$$

for s, ω chosen as above. On the other hand, if (18) holds, this right action does extend the right R-action. One has to prove that (18) can be taken as a definition of right $S^{-1}R$-action (compatible with the left action), i.e. it does not depend on choices. If we choose s', ω' such that $s'd(r) = \omega't$ then $s^{-1}\omega t = (s')^{-1}\omega't$. As t does not annihilate from the right, $s^{-1}\omega = (s')^{-1}\omega'$. Other cases are left to the reader. Hence $S^{-1}\Omega^1(R)$ is a bimodule; its elements are of the form $s^{-1}adb$.

To prove that it is sufficient, define d_S from d by the generalization of formula (15) by $\bar{\partial}$ and ∂ replaced by d_S and d and proceed with the rest of the proof as in **13.1** – all the calculations there make sense.

14 Gabriel filter $\mathcal{L}_S$ for any $S \subset R$

14.1 A **lattice** is a poset $(W, \succ)$ such that for any two elements z_1, z_2 the least upper bound $z_1 \vee z_2$ and the greatest lower bound $z_1 \wedge z_2$ exist. In other words, the binary operations *meet* $\wedge$ and *join* $\vee$ are everywhere defined. A poset is **bounded** if it contains a maximum and a minimum element, which we denote 1 and 0 respectively. A ('proper') **filter** in a bounded lattice $(W, \succ)$ is a subset $\mathcal{L} \subset W$ such that $1 \in \mathcal{L}$, $0 \notin \mathcal{L}$, $(z_1, z_2 \in \mathcal{L} \Rightarrow z_1 \wedge z_2 \in \mathcal{L})$ and $(z \in \mathcal{L}, z' \succ z \Rightarrow z' \in \mathcal{L})$.

E.g. in any bounded lattice $(W, \succ)$, given $m \in W$, the set ^mW of all $n \succ m$ is a filter.

14.2 *Notation.* Given a left ideal $J \in I_l R$ and a subset $w \subset R$ define

$$\boxed{(J : w) := \{z \in R \,|\, zw \subset J\}}$$

Then $(J : w)$ is a left ideal in R. If $w =: K$ is also a left ideal, then $(J : K)$ is 2-sided ideal. In particular, if $w = K = R$, then $(J : R)$ is the maximal 2-sided ideal contained in J. For $r \in R$ we write $(J : r)$ for $(J, \{r\})$.

Given subsets $v, w \subset R$, set $((J : v) : w)$ contains precisely all t_1 such that $t_1 w \subset (J : v)$, i.e. $t_1 wv \subset J$. Hence $((J : v) : w) = (J : wv)$.

14.3 Preorders on left ideals. Let $I_l R$ be the set of all left ideals in a ring R. It is naturally a preorder category with respect to the inclusion preorder. This category is a lattice. For the localization and spectral questions another partial order $\succ$ on $I_l R$ is sometimes better: $K \succ J$ (category notation: $J \to K$) iff either $J \subset K$, or there exist a finite subset $w \subset R$ such that $(J : w) \subset K$. Any filter in $(I_l R, \succ)$ is called a **uniform filter**.

14.4 Let R be a unital ring and $S \subset R$ a multiplicative set. Consider

$$\mathcal{L}_S := \{J \text{ left ideal in } R \,|\, \forall r, \, (J : r) \cap S \neq \emptyset\} \subset I_l R. \tag{19}$$

We make the following observations:

- As $(R : r) = R$, $R \in \mathcal{L}_S$.

- Suppose $J, K \in \mathcal{L}_S$. Given $r \in R$, $\exists s, t$, such that $s \in (J : r) \cap S$ and $t \in (K : sr) \cap S$. Hence $tsr \in J \cap K$. Set S is multiplicative, hence $ts \in S$ and $ts \in (J : r) \cap (K : r) \cap S = (J \cap K : r) \cap S$. Thus $J \cap K \in \mathcal{L}_S$.

- $(J : r) \cap S \neq \emptyset$ then, a fortiori, $(K : r) \cap S \neq \emptyset$ for $K \supset J$.

- If $J \in \mathcal{L}_S$ then $\forall r \ (J : r) \cap S \neq \emptyset$. In particular, this holds with r replaced by rr'. Using $((J : r) : r') = (J : r'r)$ we see that $(J : r) \in \mathcal{L}_S$ for all $r \in R$.

- If $\forall r' \in R \ (J : r') \cap S \neq \emptyset$ and $((J' : j) : r) \cap S \neq \emptyset$ for all $j \in J$, $r \in R$, then $\exists s \in S$ such that $srj \in J'$ and $\exists s' \in S$ such that $s'r' \in J$. In particular for $r = 1$ and $j = s'r'$ we have $ss'r' \in J$. Now $ss' \in S$ and r' is arbitrary so $J' \in \mathcal{L}_S$.

These properties can be restated as the axioms for a **Gabriel filter** $\mathcal{L} \subset I_l R$ (synonyms "radical set", "radical filter", "idempotent topologizing filter"):

- (F1) $R \in \mathcal{L}$ and $\emptyset \notin \mathcal{L}$.

- (F2) If $J, K \in \mathcal{L}$, then $J \cap K \in \mathcal{L}$.

- (F3) If $J \in \mathcal{L}$ and $J \subset K$ then $K \in \mathcal{L}$.

- (UF) $J \in \mathcal{L} \Leftrightarrow (\forall r \in R, \ (J : r) \in \mathcal{L})$.

- (GF) If $J \in \mathcal{L}$ and $\forall j \in J$ the left ideal $(J' : j) \in \mathcal{L}$, then $J' \in \mathcal{L}_S$.

Axioms (F1-3) simply say that a set $\mathcal{L}$ of ideals in R is a filter in $(I_l R, \subset)$. Together with (UF) they exhaust the axioms for a uniform filter (cf. **14.3**). Axioms (GF) and (UF) imply (F2): If $j \in J$, $(I \cap J : j) = (I : j) \cap (J : j) = (I : j) \in \mathcal{L}$ by (UF). Since $\forall j \in J$ $(I \cap J : j) \in \mathcal{L}$, (GF) implies $I \cap J \in \mathcal{L}$. (GF) & (F1) imply (F3): $(\forall j \in J \subset K)$ $(K : j) = R \in \mathcal{L}$, hence $K \in \mathcal{L}$.

There are examples of Gabriel filters $\mathcal{L}$, even for commutative R, which are *not* of the form $\mathcal{L}_S$ for a multiplicative $S \subset R$. Moreover, for rings without unity (F1-3,UF,GF) still make sense, whence a good notion of a multiplicative set and filters $\mathcal{L}_S$ fails to exist.

Notice that if a multiplicative set S satisfies the left Ore condition, then $\mathcal{L}_S = \mathcal{L}'_S := \{J$ is left ideal $\mid J \cap S \neq \emptyset\}$. Namely, $(J : 1) \cap S = J \cap S$ for *any* S, hence $\mathcal{L}_S \subset \mathcal{L}'_S$; and the *left* Ore condition implies that given an element $s \in J \cap S$ and $r \in R$ we can find $s' \in S$, and $r' \in R$ with $s'r = r's \in r'J \subset J$, hence $s' \in (J : r) \cap S$; hence $\mathcal{L}'_S \subset \mathcal{L}_S$.

14.5 Exercise. Check that the intersection of any family of Gabriel filters is a Gabriel filter.

Remark: this is *not* always true for the union: (GF) often fails.

14.6 For given R-module M and a filter $\mathcal{L}$ in $(I_l R, \subset)$, the inclusions $J \hookrightarrow J'$ induce maps $\mathrm{Hom}_R(J', M) \to \mathrm{Hom}_R(J, M)$ for any M, hence we obtain an inductive system of abelian groups. The inclusion also induce the projections $R/J \to R/J'$ and hence, by composition, the maps $\mathrm{Hom}_R(R/J', M) \to \mathrm{Hom}_R(R/J, M)$. This gives another inductive system of abelian groups. If a filter $\mathcal{L}$ is uniform, we consider the same systems and limits of groups (without new morphisms), and use (UF) as ingenious device to define the R-module structure on them.

14.7 Proposition. *Let $\mathcal{L}$ be a uniform filter and M a left R-module.*

(i) The inductive limit of abelian groups taken over downwards directed family of ideals

$$H_{\mathcal{L}}(M) := \lim_{J \in \mathcal{L}} \mathrm{Hom}_R(J, M)$$

has a canonical structure of an R-module. $H_{\mathcal{L}}$ extends to an endofunctor.

(ii) The abelian subgroup

$$\sigma_{\mathcal{L}}(M) := \{m \in M \mid \exists J \in \mathcal{L}, \ Jm = 0\} \subset M$$

is a R-submodule of M.

(iii) If $f : M \to N$ is a map of R-modules, $\mathrm{Im}\, f|_{\sigma_{\mathcal{L}}(M)} \subset \sigma_{\mathcal{L}}(N)$, hence the formula $f \mapsto \sigma_{\mathcal{L}}(f) := f|_{\sigma_{\mathcal{L}}(M)}$ extends $\sigma_{\mathcal{L}}$ to a subfunctor of identity.

(iv) The inductive limit of abelian group $\sigma'_{\mathcal{L}}(M) := \lim_{J \in \mathcal{L}} \mathrm{Hom}_R(R/J, M)$
has a structure of a left R-module.

(v) If $1 \in R$ then the endofunctors $\sigma_{\mathcal{L}}$ and $\sigma'_{\mathcal{L}}$ (on the categories of modules M with $1_R m = m$, where $m \in M$) are equivalent.

Proof. (i) Given $f \in H_{\mathcal{L}}(M)$, represent it as f_J in $\mathrm{Hom}_R(J, M)$ for some $J \in \mathcal{L}$. By (UF), $\forall r \in R$, $(J : r) \in \mathcal{L}$. The rule $x \mapsto f_J(xr)$ defines a map $(rf)_{(J:r)}$ in $\mathrm{Hom}_R((J : r), M)$ which we would like to represent class rf. Suppose we have chosen another representative f_I, then there is $K \in \mathcal{L}$, $K \subset I \cap J$, such that $f_I|_K = f_J|_K =: h$. Then $(K : r) \subset (I \cap J : r) = (I : r) \cap (J : r)$ and the map $x \mapsto h(xr) : K \to M$ agrees with $(rf)_{(J:r)}|_K$ and $(rf)_{(I:r)}|_K$ hence the class rf is well defined.

This is a left action: $((rr')f)_{(J:rr')}(x) = f_J(xrr') = (r'f)_{(J:r')}(xr) = (r(r'f))_{((J:r'):r)}(x) = (r(r'f))_{(J:rr')}(x)$. We used $((J : r') : r) = (J : rr')$.

(ii) Suppose $m \in \sigma_{\mathcal{L}}(M)$, i.e. $Jm = 0$ for some $J \in \mathcal{L}$. For arbitrary $r \in R$ the ideal $(J : r) \in \mathcal{L}$ by (UF). Let $k \in (J : r)$. Then $kr \in J$, hence $krm = 0$. This is true for any such k, hence $(J : r)rm = 0$ and $rm \in \sigma_{\mathcal{L}}(M)$. As r was arbitrary, $R\sigma_{\mathcal{L}}(M) \subset \sigma_{\mathcal{L}}(M)$.

(iii) If $m \in \sigma_{\mathcal{L}}(M)$ then $0 = f(0) = f(Jm) = Jf(m)$ for some J in $\mathcal{L}$. Hence $f(m) \in \sigma_{\mathcal{L}}(N)$.

(iv) Let $r \in R$ and $f \in \lim \mathrm{Hom}_R(R/J, M)$. Take a representative $f_J \in \mathrm{Hom}_R(R/J, M)$. Let $(rf)_{(J:r)} \in \mathrm{Hom}_R(R/(J : r), M)$ be given by $(rf)_{(J:r)}(r' + (J : r)) = f_J(r'r + J)$. This formula does not depend on r' because changing r' by an element $\delta r' \in (J : r)$ changes $r'r$ by an element $(\delta r')r$ in $(J : r)r \subset J$. Suppose $f_I \sim f_J$. In this situation, with projections as connecting morphisms, this means that $f_I(x + I) = f_J(x + J)$ for all $x \in R$, and in particular for $x = r'r$, hence $(rf)_{(J:r)} \sim (rf)_{(I:r)}$ and rf is well defined.

Finally, $f \mapsto rf$ is a left R-action. Indeed, for all $r, r', t \in R$,

$$
\begin{aligned}
((rr')f)_{(J:rr')}(t + (J : rr')) &= (f_J)(trr' + J) \\
&= (r'f)_{(J:r')}(tr + (J : r')) \\
&= (r(r'f))_{((J:r'):r)}(t + ((J : r') : r)) \\
&= (r(r'f))_{(J:rr')}(t + (J : rr'))
\end{aligned}
$$

If R and M are unital, then $1_R f = f$ as well.

(v) To make the statement precise, we should first extend $\sigma_{\mathcal{L}}$ to a functor by defining it on morphisms as well ($\sigma'_{\mathcal{L}}$ is obviously a functor as the formula on object is explicitly written in terms of a composition of functors applied on M). As $\sigma_{\mathcal{L}}(M) \subset M$, it is sufficient to show that $f(\sigma_{\mathcal{L}}(M)) = \sigma_{\mathcal{L}}(f(M))$ and then define $\sigma_{\mathcal{L}}(f) := \sigma_{\mathcal{L}} \circ f$. Element $f(m) \in f(\sigma_{\mathcal{L}}(M))$ iff $Jm = 0$ for some $J \in \mathcal{L}$. This is satisfied iff $f(Jm) = Jf(m) = 0$, i.e. $f(m) \in \sigma_{\mathcal{L}}(f(M))$.

The equivalence $\nu : \sigma'_{\mathcal{L}} \Rightarrow \sigma_{\mathcal{L}}$ is given by $\nu_M([f_J]) := f_J(1_R+J) \in \sigma_{\mathcal{L}}(M)$ (because $Jf_J(1_R + J) = f_J(0 + J) = 0$), with inverse $m \mapsto [\theta_J^{(m)}]$ where $\theta_J^{(m)} : r + J \mapsto rm$ and any $Jm = 0$ (such J exists and the formulas for $\theta_J^{(m)}$ for different J agree, hence *a fortiori* define a limit class). Starting with $[f_J]$ with $m := f_J(1_R + J)$ and $\theta_J^{(m)} : r + J \mapsto rf_J(1_R + J) = f_J(r + J)$, hence $\theta_J^{(m)} = f_J$. Other way around, start with $m \in \sigma_{\mathcal{L}}(M)$, then $\nu_M(\theta_J^{(m)}) = \theta_J^{(m)}(1_R + J) = 1_R m = m$. Hence we see that each ν_M is an isomorphism of modules. The reader may check that ν, ν^{-1} are natural transformations.

14.8 If $\mathcal{A}$ is any abelian category, then a subfunctor σ of the identity (i.e. $\sigma(M) \subset M$ and $\sigma(f)|_{\sigma(M)} = f|_{\sigma(M)}$, cf. **1.4**) with the property $\sigma(M/\sigma(M)) = 0$ is called a **preradical** in $\mathcal{A}$. A preradical σ in $R - \mathrm{Mod}$ is left exact iff $J \subset K$ implies $\sigma(J) = \sigma(K) \cap J$. A **radical** is a left exact preradical.

14.9 Proposition. *If $\mathcal{L}$ is Gabriel filter, $\sigma_{\mathcal{L}}$ is an* **idempotent radical** *in the category of left R-modules, i.e. it is a radical and $\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}$.*

14.10 To any Gabriel filter $\mathcal{L}$, one associates a localization endofunctor $Q_{\mathcal{L}}$ on the category of left modules by the formula

$$Q_{\mathcal{L}}(M) := H_{\mathcal{L}}(M/\sigma_{\mathcal{L}}(M)) = \lim_{J \in \mathcal{L}} \mathrm{Hom}_R(J, M/\sigma_{\mathcal{L}}(M)). \qquad (20)$$

Left multiplication by an element $r \in R$ defines a class $[r] \in Q_{\mathcal{L}}(R)$. There is a unique ring structure on $Q_{\mathcal{L}}(R)$, such that the correspondence $i_{\mathcal{L}} : r \mapsto [r]$ becomes a ring homomorphism $i_{\mathcal{L}} : R \to Q_{\mathcal{L}}(R)$.

Notice that (20) generalizes the RHS of Deligne's formula, **10.4a**.

14.11 Not only every Gabriel filter defines an idempotent radical, but also ([62]):

Proposition. *Every radical defines a Gabriel filter by the rule*

$$\mathcal{L}_{\sigma} := \{ J \text{ left ideal in } R \,|\, \sigma(R/J) = R/J \}.$$

More generally, if M be a left R-module and σ a radical, define

$$\mathcal{L}_{M,\sigma} := \{ L \text{ left } R\text{-submodule in } M \,|\, \sigma(M/L) = M/L \}.$$

Then $\mathcal{L}_M := \mathcal{L}_{M,\sigma}$ satisfies the following properties

- *(GT1) $M \in \mathcal{L}_M$.*

- *(GT2) If $L, K \in \mathcal{L}_M$, then $L \cap K \in \mathcal{L}$.*

- *(GT3) If $L \in \mathcal{L}_M$, $K \subset M$ a left submodule and $L \subset K$, then $K \in \mathcal{L}_M$.*

- *(GT4) If $J \in \mathcal{L}_M$ and $K \in \mathcal{L}_J$ the left submodule $K \in \mathcal{L}_M$.*

14.11a When we restrict to the *idempotent radicals*, then the rule $\sigma \mapsto \mathcal{L}_\sigma$ gives a *bijection* between the idempotent radicals and Gabriel filters.

15   Localization in abelian categories

The language of Gabriel filters is not suited for some other categories where additive localization functors are useful. Subcategories closed with respect to useful operations (e.g. extensions of objects) are often used as the localization data, particularly in abelian and triangulated categories. We confine ourselves just to a summary of basic notions in abelian setting and comment on the connection to the language of Gabriel filters, as a number of references is available ([21, 42, 43, 62, 103, 107]).

15.1 Let $\mathcal{A}$ be an additive category. Let $\mathcal{P}$ be a full subcategory of $\mathcal{A}$. Define the left and right orthogonal to $\mathcal{P}$ to be the full subcategories $^\perp\mathcal{P}$ and $\mathcal{P}^\perp$ consisting of all objects $A \in \mathcal{A}$ such that $\mathcal{A}(P, A) = 0$ (resp. $\mathcal{A}(A, P) = 0$) for all $P \in \mathcal{P}$. Zero object is the only object in $\mathcal{P} \cap {}^\perp\mathcal{P}$. It is clear that taking (left or right) orthogonal reverses inclusions and that $\mathcal{P} \subset {}^\perp(\mathcal{P}^\perp)$ and $\mathcal{P} \subset ({}^\perp\mathcal{P})^\perp$. We leave as an exercise that $\mathcal{P}^\perp = ({}^\perp(\mathcal{P}^\perp))^\perp$ and $^\perp\mathcal{P} = {}^\perp(({}^\perp\mathcal{P})^\perp)$.

15.2 A **thick subcategory** of an abelian category $\mathcal{A}$ is a strictly full subcategory $\mathcal{T}$ of $\mathcal{A}$ which is closed under extensions, subobjects and quotients. In other words, an object M' in a short sequence $0 \to M \to M' \to M'' \to 0$ in $\mathcal{A}$ belongs to $\mathcal{T}$ iff M and M'' do.

Given a pair $(\mathcal{A}, \mathcal{T})$ where $\mathcal{A}$ is abelian and $\mathcal{T} \subset \mathcal{A}$ is thick, consider the class

$$\Sigma(\mathcal{T}) := \{f \,|\, \mathrm{Ker}\, f \in \mathrm{Ob}\, \mathcal{T}, \text{ and } \mathrm{Coker}\, f \in \mathrm{Ob}\, \mathcal{T}\}.$$

The quotient category $\mathcal{A}/\mathcal{T}$ is defined as follows. $\mathrm{Ob}\,\mathcal{A}/\mathcal{T} = \mathrm{Ob}\,\mathcal{A}$ and $\mathrm{Mor}\,\mathcal{A}/\mathcal{T} := \mathrm{Mor}\,\mathcal{A} \coprod \Sigma^{-1}(\mathcal{T})$, where $\Sigma^{-1}(\mathcal{T})$ is the class of formal inverses of morphisms $f \in \Sigma$; impose the obvious relations. $\mathcal{A}/\mathcal{T}$ is additive in a unique way making the quotient functor additive. In fact ([42, 43]), it is abelian.

Proposition. (GROTHENDIECK [58]) *Let $\mathcal{T}$ be a thick subcategory in $\mathcal{A}$ and $\Sigma(\mathcal{T})$ as above. Then Σ is a left and right calculus of fractions in $\mathcal{A}$ and $\mathcal{A}[\Sigma(\mathcal{T})]^{-1}$ is naturally isomorphic to $\mathcal{A}/\mathcal{T}$.*

A thick subcategory $\mathcal{T}$ is a **localizing subcategory** if the morphisms which are *invertible* in the quotient category $\mathcal{A}/\mathcal{T}$ are precisely the images of the morphisms in $\Sigma(\mathcal{T})$.

Every exact localization functor $T^* : \mathcal{A} \to \mathcal{B}$ (i.e. an exact functor with fully faithful right adjoint T_*) of an abelian category $\mathcal{A}$ is the localization at the localizing subcategory Σ consisting of those morphisms f such that $T^* f$ is either kernel or a cokernel morphism of an invertible morphism in $\mathcal{B}$.

If $T^* : \mathcal{A} \to \mathcal{B}$ is any exact localization functor, then set $\mathcal{T} := \operatorname{Ker} T^*$ to be the full subcategory of $\mathcal{A}$ generated by all objects X such that $T^*(X) = 0$. Then T^* factors uniquely as $\mathcal{A} \xrightarrow{\mathcal{Q}^*} \mathcal{A}/\mathcal{T} \to \mathcal{B}$ where $\mathcal{Q}^*$ is the natural quotient map.

More than one thick subcategory may give the same quotient category, and that ambiguity is removed if we consider the corresponding localizing subcategories instead ([103]).

A composition of localization functors corresponds to **Gabriel multiplication** • on thick subcategories. For any two subcategories $\mathcal{B}$, $\mathcal{D}$ of an abelian category $\mathcal{A}$ one defines $D \bullet B$ to be the full subcategory of $\mathcal{A}$ consisting of precisely those A in $\mathcal{A}$ for which there is an exact sequence $0 \to B \to A \to D \to 0$ with B in $\operatorname{Ob}\mathcal{B}$ and D in $\operatorname{Ob}\mathcal{D}$. In categories of modules one can redefine Gabriel multiplication in terms of radical filters, cf. ([113]).

15.3 In this article, we often view exact localizations (and quotient categories, cf. **10.4b**) as categorical analogues of open spaces. Their complements should then be the complementary data to the quotient categories, and such data are localizing subcategories. A more precise and detailed discussion of those subcategories, which may be considered as subschemes and closed subschemes, may be found in [79], Part I and [113, 124]. Cf. the notion of a (co)reflective subcategory in **8.9**.

Thus, in our view, it is geometrically more appealing to split the data of a category to a localizing subcategory and a quotient category, than into two *sub*categories. However, the latter point of view is more traditional, under the name of "torsion theory' and has geometrically important analogues for triangulated categories. A **torsion theory** ([21, 62]) in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{A}$ closed under isomorphisms and such that $\mathcal{F}^\perp = \mathcal{T}$ and $^\perp\mathcal{T} = \mathcal{F}$.

For any idempotent radical σ in $\mathcal{A}$ (**14.9**), the class $\mathcal{T}_\sigma$ of σ-**torsion objects** and the class $\mathcal{F}_\sigma$ of σ-**torsion free objects** are defined by formulas

$$\mathcal{T}_\sigma = \{M \in \operatorname{Ob}\mathcal{A} \,|\, \sigma(M) = M\}, \quad \mathcal{F}_\sigma = \{M \in \operatorname{Ob}\mathcal{A} \,|\, \sigma(M) = 0\}.$$

This pair $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ is an example of a torsion theory and $\mathcal{T}_\sigma$ is a thick subcategory of $\mathcal{A}$. Not every torsion theory corresponds to a radical, but hereditary theories do. That means that a subobject of a torsion object is torsion. The

Cohn localization of the next section is not necessarily hereditary, but it is always a torsion theory as shown there.

16   Quasideterminants and Cohn localization

Notation. Let $M_m^n(R)$ be the set of all $n \times m$ matrices over a (noncommutative) ring R, so that $M_n(R) := M_n^n(R)$ is a ring as well. Let I, J be the ordered tuples of row and column labels of $A = (a_j^i) \in M_m^n(R)$ respectively. For subtuples $I' \subset I, J' \subset J$ and $A = (a_j^i) \in M_m^n(R)$, denote by $A_{J'}^{I'}$ the submatrix of $A_J^I := A$ consisting only of the rows and columns with included labels; e.g. $A_{\{j\}}^{\{i\}} = a_j^i$ is the entry in i-th row and j-th column. When I is known and $K \subset I$, then $|K|$ is the cardinality of K and $\hat{\ }$ is the symbol for omitting, i.e. $\hat{K} = I \backslash K$ is the complementary $(|K| - |I|)$-tuple.

We may consider the r-tuple $\tilde{I} = (I_1, \ldots, I_r)$ of sub-tuples which partitions the n-tuple $I = (i_1, i_2, \ldots, i_n)$, i.e. I_k are disjoint and all labels from I are included; then $|\tilde{I}| := r$. Given $\tilde{I}, \tilde{J}$ form the corresponding **block matrix** in $M_{\tilde{J}}^{\tilde{I}}$ out of A, i.e. the $|\tilde{I}| \times |\tilde{J}|$ matrix $A_{\tilde{J}}^{\tilde{I}}$ whose entries are matrices $A_{\tilde{J}_l}^{\tilde{I}_k} := A_{J_l}^{I_k}$ cut-out from A by choosing the selected tuples. Forgetting the partition gives the canonical bijection of sets $M_{\tilde{J}}^{\tilde{I}} \to M_J^I$. The multiplication of block matrices is defined by the usual matrix multiplication formula $(AB)_{J_j}^{I_i} = \sum_{l=1}^r A_{K_l}^{I_i} A_{J_j}^{K_l}$ if AB *and* the sizes of subtuples for columns of A and rows of B match. One can further nest many levels of partitions (block-matrices of block-matrices ...). Some considerations will not depend on whether we consider matrices in R or block matrices, and then we'll just write M_J^I etc. skipping the argument. More generally, the labels may be the objects in some abelian category $\mathcal{A}$, and entries $a_j^i \in \mathcal{A}(i, j)$; I will be the sum $\oplus_{i \in I} i$, hence $A : I \to J$. Ring multiplication is replaced by the composition, defined whenever the labels match.

Observation. *Multiplication of block matrices commutes with forgetting (one level) of block-matrix structure.* In other words we may multiply in stages (if working in $\mathcal{A}$ this is the associativity of $\oplus$). Corollaries:

(i) if $\tilde{I} = \tilde{J}$ then $M_{\tilde{I}}(R) := M_{\tilde{I}}^{\tilde{I}}(R)$ is a ring.

(ii) We can invert matrices in stages as well ('heredity').

(iii) The same for linear equations over noncommutative rings.

Any pair $(i, j) \in I \times J$ determines partitions $\tilde{I} = (i, \hat{i})$ and $\tilde{J} = (j, \hat{j})$. For each A in M_J^I it induces a 2×2 block-matrix $A_{\tilde{J}}^{\tilde{I}}$. Reader should do the exercise of inverting that block matrix (with noncommutative entries), in

terms of the inverses of blocks. As we will see, the (i, j)-quasideterminant of A is the inverse of the (j, i)-entry of A^{-1} if the latter is defined; though it may be defined when the latter is not.

16.1 The (i, j)-th quasideterminant $|A|_{ij}$ of A is

$$\boxed{|A|_{ij} = a^i_j - \sum_{k \neq i, l \neq j} a^i_l (A^{\hat{i}}_{\hat{j}})^{-1}_{lk} a^k_j} \tag{21}$$

provided the right-hand side is defined (at least in the sense of evaluating a rational expression, which will be discussed below). In alternative notation, the distinguished labels ij may be replaced by a drawing of a box around the entry a^i_j as in

$$\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & \boxed{a^3_2} & a^3_3 \end{vmatrix} \equiv \begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix}_{32}$$

At most n^2 quasideterminants of a given $A \in M_n(R)$ may be defined.

16.2 *If all the n^2 quasideterminants $|A|_{ij}$ exist and are invertible then the inverse A^{-1} of A exist in $A \in M_n(R)$ and*

$$(|A|_{ji})^{-1} = (A^{-1})^i_j. \tag{22}$$

Thus we also have

$$|A|_{ij} = a^i_j - \sum_{k \neq i, l \neq j} a^i_l |A^{\hat{i}}_{\hat{j}}|^{-1}_{kl} a^k_j \tag{23}$$

16.3 Sometimes the RHS of (21) makes sense while (23) does not. So for subtle existence questions one may want to be careful with alternative formulas for quasideterminants. Some identities are often proved using alternative forms, so one has to justify their validity. Different expressions differ up to **rational identities** ([1, 27]), and under strong assumptions on the ring R (e.g. a skewfield which is of ∞ dimension over the center which is also infinite), the rational identities induce a well-behaved equivalence on the algebra of rational expressions and the results of calculations extend in an expected way to alternative forms once they are proved for one form having a nonempty domain of definition ([1, 27, 123])

16.4 On the other hand, the existence of inverse A^{-1} does *not* imply the existence of quasideterminants. For example, the unit 2×2 matrix $\mathbf{1}_{2 \times 2}$ over field $\mathbb{Q}$ has only 2 quasideterminants, not 4. Or, worse, matrix $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

over commutative ring $\mathbb{Z}[\frac{1}{5}]$ is invertible, but no single entry is invertible, and in particular no quasideterminants exist.

16.5 Quasideterminants are *invariant under permutation* of rows or columns of A if we appropriately change the distinguished labels.

16.6 Suppose now we are given an equation of the form

$$Ax = \xi$$

where $A \in M_n(R)$ and x, ξ are n-tuples of indeterminates and free coefficients in R respectively (they are column "vectors"). Then one can attempt to solve the system by finding the inverse of matrix A and multiply the equation by A^{-1} from the left, or one can generalize the Cramer's rule to the noncommutative setup.

Define thus $A(j, \xi)$ as the $n \times n$ matrix whose entries are the same as of A except that the j-th column is replaced by ξ. Then the noncommutative **left Cramer's rule** says

$$\boxed{|A|_{ij} x^j = |A(j, \xi)|_{ij}}$$

and the right-hand side does not depend on i.

To see that consider first $n = 2$ case:

$$a_1^1 x^1 + a_2^1 x^2 = \xi^1$$
$$a_1^2 x^1 + a_2^2 x^2 = \xi^2$$

Then

$$
\begin{aligned}
|A|_{11} x^1 &= a_1^1 x^1 - a_2^1 (a_2^2)^{-1} a_1^2 x^1 \\
&= (\xi^1 - a_2^1 x^2) - a_2^1 (a_2^2)^{-1} a_1^2 x^1 \\
&= \xi^1 - a_2^1 (a_2^2)^{-1} a_2^2 x^2 - a_2^1 (a_2^2)^{-1} a_1^2 x^1 \\
&= \xi^1 - a_2^1 (a_2^2)^{-1} \xi^2 = |A(1, \xi)|_{11}.
\end{aligned}
$$

The general proof is exactly the same, just one has to understand which indices are included or omitted in the sums involved:

$$
\begin{aligned}
|A|_{ij} x^j &= a_j^i x^j - \sum_{k \neq j, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_j^l x^j \\
&= (\xi^i - \sum_{h \neq j} a_h^i x^h) - \sum_{k \neq j, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_j^l x^j \\
&= \xi^i - \sum_{h \neq j, k \neq j, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_h^l x^h - \sum_{k \neq j, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_j^l x^j \\
&= \xi^i - \sum_{1 \leq h \leq n, k \neq j, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_h^l x^h \\
&= \xi^i - \sum_{1 \leq h \leq n, l \neq i} a_k^i (A_{\hat{j}}^{\hat{i}})_{kl}^{-1} a_h^l x^h \\
&= |A(j, \xi)|_{ij}.
\end{aligned}
$$

Similarly consider equation $\sum_k y^k B_k^l = \zeta^l$. Apparently the individual coefficients multiply y^k from the right, but the combinatorics of matrix labels is organized as if we multiply By (alas, otherwise the rule of writing upper indices for rows would force us to write such equations upside-down!). The canonical antiisomorphism $R \to R^{\mathrm{op}}$ clearly sends any quasideterminant into the quasideterminant of the *transposed matrix*. Hence the left Cramer's rule implies the **right Cramer's rule**

$$\boxed{y^j |B^T|_{ji} = |(B(j,\zeta))^T|_{ji}.}$$

16.7 Row and column operations. Ordinary determinants do not change if we add a multiple of one row to another, and similarly for the columns.

We have to distinguish between left and right linear combinations.

If $|A|_{ij}$ is defined and $i \neq l$, then it is unchanged under left-row operation

$$A^l \to A^l + \sum_{s \neq l} \lambda_s A^s$$

Proof. We may assume $i = 1$. Define the row matrix

$$\vec{\lambda} = (\lambda_2, \ldots, \lambda_n).$$

Then $\vec{\lambda}T = \sum_{s \neq k} \lambda_s T^s$ for any matrix T with row-labels $s = 2, \ldots, n$. Then $\Lambda T = \sum_{s \neq k} \lambda_s T^s$. Assume the matrix A is in the block-form written as

$$\begin{pmatrix} a & \vec{b} \\ \vec{c}^T & D \end{pmatrix}$$

with a of size 1×1. Then

$$\begin{vmatrix} \boxed{a + \vec{\lambda}\vec{c}^T} & \vec{b} + \vec{\lambda}D \\ \vec{c}^T & D \end{vmatrix} \;=\; a + \vec{\lambda}\vec{c}^T - (\vec{b} + \vec{\lambda}D)\,D^{-1}\,\vec{c}^T$$

$$=\; a - \vec{b}D^{-1}\vec{c}^T.$$

If we multiply the l-th row from the left by an invertible element μ then the quasideterminant $|A|_{ij}$ won't change for $i \neq l$ and will be multiplied from the left by μ if $i = l$. Actually, more generally, left multiply the i-th row by μ and the block matrix consisting of other rows by invertible square matrix Λ (i.e. other rows can mix among themselves, and scale by different factors):

$$A \to \begin{pmatrix} \mu & 0 \\ 0 & \Lambda \end{pmatrix}$$

Then $|A|_{ij}$ gets left-multiplied by μ:

$$\left| \begin{array}{cc} \boxed{\mu a} & \mu\vec{b} \\ \Lambda\vec{c}^T & \Lambda D \end{array} \right|_{11} \begin{array}{l} = & \mu a - \mu\vec{b}\,(\Lambda D)^{-1}\,\Lambda\vec{c}^T \\[2mm] = & \mu\,(a - \vec{b}D^{-1}\vec{c}^T) = \mu\,|A|_{ij}. \end{array}$$

16.8 Jacobi's ratio theorem. ([46]) *Let A be a matrix with possibly noncommutative entries such that the inverse $B = A^{-1}$ is defined. Choose some row index i and some column index j. Make a partition of the set of row indices as $I \cup \{i\} \cup J$ and a partition of the set of column indices as $I' \cup \{j\} \cup J'$, with the requirements $\operatorname{card} I = \operatorname{card} I'$ and $\operatorname{card} J = \operatorname{card} J'$. Then*

$$(|A_{I\cup\{i\},I'\cup\{j\}}|_{ij})^{-1} = |B_{J'\cup\{j\},J\cup\{i\}}|_{ji}.$$

Proof. ([71]) The block decomposition of matrices does not change the multiplication, i.e. we can multiply the block matrices and then write out the block entries in detail, or we can write the block entries of the multiplicands in detail and then multiply and we get the same result. In particular, as $A = B^{-1}$, the block-entries of A can be obtained by block-inversion of B.

After possible permutation of labels, we may find the block-entry of the matrix $A = B^{-1}$ at the intersection of rows $I \cup \{i\}$ and columns $I' \cup \{j\}$ by means of block-inverting the block matrix

$$A = \left(\begin{array}{cc} A_{I\cup\{i\},I'\cup\{j\}} & A_{I\cup\{i\},J'} \\ A_{J,I'\cup\{j\}} & A_{J,J'} \end{array} \right)$$

Then $A_{I\cup\{i\},I'\cup\{j\}} = (B_{I\cup\{i\},I'\cup\{j\}} - B_{I\cup\{i\},J'}(B_{JJ'})^{-1}B_{J,I\cup\{i\}})^{-1}$ or, equivalently,

$$(A_{I\cup\{i\},I'\cup\{j\}})^{-1} = B_{I\cup\{i\},I'\cup\{j\}} - B_{I\cup\{i\},J'}(B_{JJ'})^{-1}B_{J,I'\cup\{j\}}$$

This is a matrix equality, and therefore it implies the equality of the (i,j)-th entry of both sides of the equation. We obtain

$$((A_{I\cup\{i\},I'\cup\{j\}})^{-1})_{ij} = b_{ij} - \sum_{k\in J',l\in J} b_{i,k}(B_{JJ'})^{-1}_{kl}b_{lj}.$$

Finish by applying the formula $(|C|_{ji})^{-1} = (C^{-1})_{ij}$ at LHS.

16.9 Muir's law of extensionality. ([46, 48, 49]) Let an identity $\mathcal{I}$ between quasiminors of a submatrix A^I_J of a generic matrix A be given. Let $K \cap I = \emptyset$, $L \cap J = \emptyset$ and $K = L$. If every quasiminor $|A^U_V|_{uv}$ of A^I_J in the

identity $\mathcal{I}$ is replaced by the quasiminor $|A_{V \cup L}^{U \cup K}|_{uv}$ of $A_{J \cup L}^{I \cup K}$ then we obtain a new identity $\mathcal{I}'$ called the extensional to $\mathcal{I}$.

16.10 Quasitelescoping sum. Let $A = (a_j^i)$ be a generic $n \times n$ matrix. For any $k > 2$, and $i, j \in \{1, k-1\}$ consider the quasiminor

$$|A_{j,k,k+1,\dots,n}^{i,k,k+1,\dots,n}|_{ij}.$$

The quasitelescoping sum involves such minors:

$$QT(A_{1,\dots,n}^{1,\dots,n}) = \sum_{k=3}^{n} |A_{k-1,k,\dots,n}^{1,k,\dots,n}|_{1,k-1} |A_{k-1,k,\dots,n}^{k-1,k,\dots,n}|_{k-1,k-1}^{-1} |A_{1,k,\dots,n}^{k-1,k,\dots,n}|_{k-1,1}$$

Then, by Muir's law and induction on n, we obtain

$$QT(A_{1,\dots,n}^{1,\dots,n}) = a_1^1 - |A|_{11}. \tag{24}$$

For $n = 3$ this is simply the identity obtained by extending by the third row and column the identity expressing the expansion of the 2×2 upper left quasiminor. Suppose now we have proved (24) for n. Take an $(n+1) \times (n+1)$-matrix A. Then, by induction, this is true for the submatrix

$$A_{\hat{2}}^{\hat{2}} = A_{1,3,\dots,n}^{1,3,\dots,n}.$$

But

$$\begin{aligned}
QT(A_{1,\dots,n}^{1,\dots,n}) &= QT(A_{1,3,\dots,n}^{1,3,\dots,n}) + |A_{2,3,\dots,n}^{1,3,\dots,n}|_{1,2} |A_{2,3,\dots,n}^{2,3,\dots,n}|_{2,2}^{-1} |A_{1,3,\dots,n}^{2,3,\dots,n}|_{2,1} \\
&= a_1^1 - |A_{1,3,\dots,n}^{1,3,\dots,n}|_{11} + |A_{2,3,\dots,n}^{1,3,\dots,n}|_{1,k-1} |A_{2,3,\dots,n}^{2,3,\dots,n}|_{k-1,k-1}^{-1} |A_{1,3,\dots,n}^{2,3,\dots,n}|_{k-1,1} \\
&= a_1^1 - |A_{1,2,3,\dots,n}^{1,2,3,\dots,n}|_{11}
\end{aligned}$$

where the last two summands were added up, using the identity which represents the expansion of 2×2 upper left corner of A and extending the identity by rows and columns having labels $3, \dots, n$.

16.11 Homological relations. Start with the identity

$$(a_1^1 - a_2^1 (a_2^2)^{-1} a_1^2)(a_1^2)^{-1} = -(a_2^1 - a_1^1 (a_1^2)^{-1} a_2^2)(a_2^2)^{-1}.$$

which in the quasideterminant language reads

$$\begin{vmatrix} \boxed{a_1^1} & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} (a_1^2)^{-1} = - \begin{vmatrix} \boxed{a_1^1} & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} (a_1^2)^{-1}$$

and extend the latter applying Muir's law, adding the rows $3, \ldots, n$ of A to each minor in the expression. Renaming the indices arbitrarily we obtain the **row homological relations**:

$$\boxed{|A|_{ij}|A^{\hat{i}}_{\hat{j}'}|^{-1}_{i'j} = -|A|_{ij'}|A^{\hat{i}}_{\hat{j}}|^{-1}_{i'j'}}$$
(25)

for $j \neq j'$. Similarly, starting with the identity

$$(a^1_2)^{-1}(a^1_1 - a^1_2(a^2_2)^{-1}a^2_1) - (a^2_2)^{-1}(a^2_1 - a^2_2(a^2_2)^{-1}a^2_1),$$

we obtain the **column homological relations**

$$\boxed{|A^{\hat{i}'}_{\hat{j}}|^{-1}_{ij'}|A|_{ij} = -|A^{\hat{i}}_{\hat{j}}|^{-1}_{i'j'}|A|_{i'j}.}$$
(26)

16.12 Laplace expansion for quasideterminants. Start with the identity

$$\sum_j a^i_j (A^{-1})^j_k = \delta^i_k.$$

If $i \neq k$ and A^{-1} exists, then substituting $(A^{-1})^j_k = |A|^{-1}_{kj}$ this becomes

$$\sum_j a^i_j |A|^{-1}_{ij} = 1.$$

Multiply this equation from the right by $|A|_{il}$ and split the sum into the part with $j \neq l$ and the remaining term:

$$a^i_l + \sum_{j \neq l} a^i_j |A|^{-1}_{ij} |A|_{il} = |A|_{il}$$

and apply the row homological relations (25) to obtain the following Laplace expansion for the (i, j)-th quasideterminant by the k-th row:

$$\boxed{a^i_l - \sum_{j \neq l} a^i_j |A^{\hat{i}}_{\hat{l}}|^{-1}_{kj} |A^{\hat{i}}_{\hat{j}}|_{kl} = |A|_{il}}$$
(27)

Similarly, multiplying from the left the equation $\sum_i |A|^{-1}_{ij} a^i_j = 1$ by $|A|_{lj}$ and splitting the sum into two terms we obtain

$$a^l_j + \sum_{i \neq l} |A|_{lj} |A|^{-1}_{ij} a^i_j = |A|_{lj},$$

which after the application of the column homological relations (26) gives the following Laplace expansion for the (i,j)-th quasideterminant by the k-th column:

$$a_j^l - \sum_{j\neq l} |A_{\hat{j}}^{\hat{i}}|_{lk} |A_{\hat{j}}^{\hat{l}}|_{ik}^{-1} a_l^i = |A|_{lj} \qquad (28)$$

Notice that the summation sign involves $(n-1)$ summands whereas the similar summation in the recursive formula (23) for quasideterminants involves $(n-1)^2$ summands.

16.13 ([27, 28]) Let R be an associative unital ring, and Σ a given set of square *matrices* of possibly different (finite) sizes with entries in R. Map $f : R \to S$ of rings is Σ-inverting if each matrix in Σ is mapped to an invertible matrix over S. A Σ-inverting ring map $i_\Sigma : R \to R_\Sigma$ is called **Cohn localization** (or universal Σ-inverting localization) if for every Σ-inverting ring map $f : R \to S$ there exist a unique ring map $\tilde{f} : R_\Sigma \to S$ such that $f = \tilde{f} \circ i_\Sigma$.

A set Σ of matrices is called (upper) **multiplicative** if $1 \in \Sigma$ and, for any $A, B \in \Sigma$ and C of right size over R, $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is in Σ. If Σ is the smallest multiplicative set of matrices containing Σ_0, then a map is Σ_0-inverting iff it is Σ-inverting. Inclusion $\Sigma_0 \subset \Sigma$ makes every Σ-inverting map $f : R \to S$ also Σ_0-inverting. Conversely, if each of the diagonal blocks can be inverted, a block-triangular matrix can be inverted, hence Σ_0-inverting maps are Σ-inverting.

The universal Σ-inverting localization can be constructed by "invertive method", as follows. Represent R as a free algebra F on a generating set $\mathbf{f}$ modulo a set of relations I. For each quadratic matrix $A \in \Sigma$ of size $n \times n$, add n^2 generators (A, i, j) to $\mathbf{f}$. This way we obtain a free algebra F' over some generating set $\mathbf{f}'$. All (A, i, j) for fixed A clearly form a $n \times n$-matrix A' over F'. Then $\Sigma^{-1}R = F'/I'$ where I' is the ideal generated by I and by all elements of matrices $AA' - I$ and $A'A - I$ for all $A \in \Sigma$. Then $i_\Sigma : R \to \Sigma^{-1}R$ is the unique map which lifts to the embedding $F \hookrightarrow F'$.

16.14 Warning. A naive approach to quotient rings, would be just adding new generators a' and relations $aa' = a'a = 1$ for each $a \in R \backslash \{0\}$ which needs to be inverted in the first place. In geometrical applications this could induce pretty unpredictable behaviour on modules etc. But suppose we just want to do this in an extreme special case: constructing a quotient skewfield. After inverting all the nonzero elements, we try inverting all their nonzero sums and so on. The problem arises that one may not know which elements from m-th step will be forced to zero by new relations added a few steps later. So one should skip inverting some new elements, as they will

become zeros after a few more steps of inverting other elements. There is no recipe which elements to leave out at each step. For a given ring R, there may be none (no quotient field) or multiple possibilities for such a recipe. More precisely, given two embeddings $R \hookrightarrow K_i$ into skewfields $K_1 \neq K_2$, there may be *different* smallest subskewfields $L_i \hookrightarrow K_i$ containing R.

16.15 Proposition. *Let Σ be multiplicative set of square matrices over R and $f : R \to S$ a Σ-inverting map. Let $S(i, \Sigma) \subset R$ consists of all components of solutions over S of all equations $f(A)x = f(b)$ where $A \in \Sigma$, b is a column-vector over R and x a column of unknowns.*

(i) $S(i, \Sigma)$ is a subring of S.

(ii) $S(i, \Sigma)$ coincides with the image of R_Σ under the unique map $\tilde{f} : R_\Sigma \to S$ for which $f = \tilde{f} \circ i_\Sigma$.

In particular, if f is 1-1 then $\tilde{f}$ is isomorphism and i_Σ is 1-1.

(i) If components x_i and y_j of column vectors x and y over S are in $S(i, \Sigma)$, with $f(A)x = b$ and $f(B)y = c$, then, possibly after enlarging x, y, b, c by zeroes and A and B by diagonal unit blocks, we may always make $i = j$ and b and c of the same length. Then $f \begin{pmatrix} A & -A+B \\ 0 & B \end{pmatrix} \begin{pmatrix} x+y \\ y \end{pmatrix} = f \begin{pmatrix} b+c \\ c \end{pmatrix}$ and as the left-hand side matrix is in $f(\Sigma)$ by multiplicativity, then $x_i + y_i \in S(i, \Sigma)$, as claimed. For z a (row or column) vector consider the diagonal square matrix $\mathrm{diag}(z)$ with diagonal z. Then $\mathrm{diag}(z)(1, 1, \ldots, 1)^T = (z_1, \ldots, z_n)$. For a fixed i, there is a matrix P_i such that $P_i(y_1, \ldots, y_n)^T = (y_i, \ldots, y_i)^T$. Hence, $f \begin{pmatrix} B & -\mathrm{diag}(c)P_i \\ 0 & A \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = f \begin{pmatrix} 0 \\ b \end{pmatrix}$ has as the j-th component $(f(B)^{-1}f(c))_j (f(A)^{-1}f(b))_i$. But our block-triangular matrix is in Σ, hence $x_i y_j$ is in $S(i, \Sigma)$. Similarly, had we worked with algebras over $\mathbf{k}$, we could have considered all possible weights on the diagonal instead of just using the non-weighted diagonal $\mathrm{diag}(c)$ to obtain any possible $\mathbf{k}$-linear combination of such.

(ii) The corestriction of i onto $S(i, \Sigma)$ is also Σ-inverting. Hence there is a unique map form R_Σ. But, by construction, there is no smaller ring than $S(i, \Sigma)$ containing $f(R_\Sigma)$. As $i(R_\Sigma)$ is a ring they must coincide. If the map is 1-1 it has no kernel hence $\tilde{f}$ is an isomorphism.

16.16 Proposition. (left-module variant of P. M. Cohn [28], 2.1) *If Σ is multiplicative, then $\exists!$ subfunctor $\sigma_\Sigma : R - \mathrm{Mod} \to R - \mathrm{Mod}$ of identity such that, as a subset, $\sigma_\Sigma(M)$ equals*

$$\{m \in M \,|\, \exists u = (u_1, \ldots, u_n)^T \in M^{\times n}, \exists i, \, m = u_i \text{ and } \exists A \in \Sigma, \, Au = 0\}$$

for every $M \in R - \mathrm{Mod}$. Moreover, σ_Σ is an idempotent preradical.

Proof. 1. $\sigma_\Sigma(M)$ *is an R-submodule of M.* It is sufficient to show that for any $r \in R$, $m, m' \in \sigma_\Sigma(M)$ the left linear combination $m + rm' \in \sigma_\Sigma(M)$. Choose $A, B \in \Sigma$, $Au = 0, Bv = 0$, $u \in M^{\times k}, v \in M^{\times l}$, $m = u_i, m' = v_j$. We may assume $k = l$, $i = j$, hence $m + rm' = (u + rv)_i$, by adjusting A, B, u, v. For example, $\tilde{A} := \mathrm{diag}(I_s, A, I_t) \in \Sigma$ and $\tilde{u} := (0_s, u_1, \ldots, u_k, 0_t)^T \in M^{s+k+t}$ satisfy $\tilde{A}\tilde{u} = 0$ with $m = \tilde{u}_{i+s}$.
Then $\begin{pmatrix} A & -Ar \\ 0 & B \end{pmatrix} \in \Sigma$ and $\begin{pmatrix} A & -Ar \\ 0 & B \end{pmatrix} \begin{pmatrix} u + rv \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2. $M \mapsto \sigma_\Sigma(M)$ *extends to a unique subfunctor of identity.* If $m = u_i \in \Sigma$ for some i and $A(u_1, \ldots, u_k)^T = 0$ then $A(f(u_1), \ldots, f(u_k))^T = 0$ whenever $f : M \to N$ is R-module map. As $f(m) = f(u_i)$ this proves that $f(\sigma_\Sigma(M)) \subset \sigma_\Sigma(f(M))$ as required.

3. $\sigma_\Sigma(M)$ *is a preradical:* $\sigma_\Sigma(M/\sigma_\Sigma(M)) = 0$. If $m \in \sigma_\Sigma(M)$, then $\exists u_1, \ldots, u_k \in M$, $\exists A \in \Sigma$, $\exists p \leq k$, such that $A\vec{u}^T = 0 \mod \sigma_\Sigma(M)$ and $m = u_p$ where $\vec{u} := (u_1, \ldots, u_k)$. Hence $\exists v_1, \ldots, v_k \in \sigma_\Sigma(M)$ such that $A\vec{u}^T = (v_1, \ldots, v_k)^T$ and there are matrices $B_1, \ldots, B_k$, where B_s is of size $h_s \times h_s$, and vectors $(w_{1s}, \ldots, w_{h_s s})$ of size h_s, such that $B_s(w_{1s}, \ldots, w_{hs})^T = 0$ for all s; and we have that $v_i = w_{s_i i}$ for some correspondence $i \mapsto s_i$. Let $\vec{w} = (w_{11}, \ldots, w_{h_1 1}, w_{12}, \ldots, w_{k s_k})$. Let matrix $J = (J^i_j)$ be defined by $J^i_{s_i} = 1$ for each i and all other entries are 0. This matrix by construction satisfies $J\vec{w}^T = \vec{v}$. Define also the block matrix $B := \mathrm{diag}\{B_1, \ldots, B_k\}$. Clearly $B\vec{w}^T = 0$ by construction and $B \in \Sigma$ by multiplicativity of Σ. In this notation the summary of just said is encoded in this block identity

$$\begin{pmatrix} A & -J \\ 0 & B \end{pmatrix} (\vec{u}, \vec{v})^T = 0, \quad \begin{pmatrix} A & -J \\ 0 & B \end{pmatrix} \in \Sigma, \quad m = u_p.$$

4. $\sigma_\Sigma(\sigma_\Sigma(M)) = \sigma_\Sigma(M)$. If $m = u_i$ for some i and $A(u_1, \ldots, u_k)^T = 0$ for some $A \in \Sigma$ with all $u_j \in \sigma_\Sigma(M)$, then in particular, all $u_j \in M$.

Exercise. *Let Σ, Σ' be multiplicative sets of matrices over R.*

If for every $A \in \Sigma$ there are permutation matrices $w, w' \in GL(k, \mathbb{Z})$ such that $wAw' \in \Sigma'$ then $\sigma_{\Sigma'}(M) \subset \sigma_\Sigma(M)$ for all M.

16.17 Warning. σ_Σ is not necessarily left exact. Equivalently, the associated torsion theory is not always hereditary (i.e. a submodule of a σ_Σ-torsion module is not necessarily σ_Σ-torsion). Hereditary torsion theories correspond to Gabriel localizations.

16.18 Quasideterminants vs. Cohn localization. Quasidetermi-nants are given by explicit formulas. It is sometimes more algorithmically manageable to invert them, than the matrices (if the inverse can not be

expressed in terms of them anyway). The two procedures often disagree, as some simple (e.g. diagonal) matrices do not possess some among n^2 possible quasideterminants. One may combine the process, by first inverting quasideterminants which exist, and then performing the Cohn localization for the simpler matrix so obtained. The combination is not necessarily a Cohn localization.

Thus, let Σ be as before. For each $A \in \Sigma$ and pair (ij) such that $|A|_{ij}$ exists and is nonzero add a variable B_{ij} and require $B_{ij}|A|_{ij} = |A|_{ij}B_{ij} = 1$. One obtains a localization $j : R \to R_{\Sigma}^{q0}$. Then one inverts $j(\Sigma)$ by the Cohn method, which amounts to adding formal variables just for those entries which are not added before as quasideterminants, and adding relations for them. The result is some localization $i_{\Sigma}^{q} : R \to R_{\Sigma}^{q}$ which is Σ-inverting and clearly a quotient ring of the Cohn localization. If i_{Σ}^{q} is injective, it is just the Cohn localization.

There are obvious variants of this method, (cf. reasoning in **16.14**, and recall that quasideterminants may be defined inductively by size). Some rings may be quotiented by ideals to get commutative or Ore domains. A quasideterminant may be proven to be nonzero, as its image in the quotient is nonzero, which is a good procedure for some concrete Σ (cf.[122], Th.7).

We have seen in Ch.8 that for the usual descent of quasicoherent sheaves one needs flatness, which is often lacking for Cohn localization. In the special case of Cohn localization at a 2-sided ideal, the flatness of the localization map i_{Σ} as a map of left modules is equivalent to the right Ore condition. Though in geometrical situations one inverts sets of matrices for which this theorem does not apply, flatness is not expected for useful non-Ore universal localizations. Less essential, but practically difficult, is to find the kernel of the localization map i_{Σ} (there is a criterium using the normal form mentioned below).

We would still like, in the spirit of an example [122], Th.7 (more accurately described in [123]), to be able to consider some global noncommutative spaces where the local coordinates are compared using nonflat Cohn localizations. Localizations between full categories of modules ('perfect localizations') are described by their underlying rings (the forgetful functor from the localized category to the category of modules over localized ring is an equivalence). Similarly, the knowledge of the restriction of the Cohn localization functor to the category of finitely generated (f.g.) projective modules is equivalent to the knowledge of the localization morphism on the level of rings. Of course, the theory is here not any more complicated if one inverts any multiplicative set of morphisms between f.g. projectives than only ma-

trices. The localization functor for f.g. projectives has the explicit descrip-
tion (Gerasimov-Malcolmson normal form ([50, 86])) (analogous to the
description of Ore localization as $S \times R/\sim$ where $\sim$ is from **4.13**) and has
an interesting homological interpretation and properties ([97]). Thus while
the torsion theory corresponding to Cohn localization is bad (nonhereditary,
cf. **10.3**,**15.3**,**16.17**), other aspects are close to perfect localizations (thus
better than arbitrary hereditary torsion theory). This suggests a hope in
a geometry of "covers by Cohn localizations" if we learn a way beyond the
case of flat descent for full categories of modules.

Bibliography. We try to give a few of the most useful references for a reader who
has similar geometric needs to the author. The bibliography is intentionally incomplete.
For the benefit of the geometrically oriented reader, we used the following preference
criteria: geometrically motivated, historically important, readable (for the author at least),
irreplacable. The literature which is obscure to me in a major way is naturally not
in the list. However I mention here some undoubtedly important alternative works by
listing only Math. Reviews code. For ring theorists there is a monograph on torsion
theories by Golan MR88c:16934 and on localization by Jategaonkar MR88c:16005 and
by Golan MR51:3207. Many equivalent approaches to Gabriel localization have been
multiply discovered (Goldman (1969) [53], Silver (1967) MR36:205, Maranda (1964)
MR29:1236 etc.) in various formalisms, e.g. torsion theories (the term is basically from
Dickson (1965) MR32:2472).

Despite their historical importance, we ignore these, and recommend the system-
atic treatment in Gabriel's thesis [42] as well as the books [62, 103, 130] and Ch. 6
of [23]. For abelian categories see [21, 39, 42, 44, 45, 62, 103, 125, 58, 144]; for lo-
calization in abelian categories see books [21, 39, 62, 103, 125]. Other longer bibli-
ographies on localization are in [39, 62, 72, 103]. Neither the present article nor the
bibliography survey noncommutative geometry beyond the localization aspects; rather
consult [26, 30, 55, 74, 75, 89, 113, 117, 125, 127, 135, 138, 140] and bibliographies
therein; quantum group literature (e.g. [83, 87]; Chari, Pressley MR95j:17010; Ma-
jid MR90k:16008, MR97g:17016, MR2003f:17014; Klimyk-Schmüdgen MR99f:17017;
Várilly MR2004g:58006); and, for the physics, also [33]. Abbrev.: LMS = London Math-
ematical Society, MPI = Max Planck Inst. preprint (Bonn). & for Springer series: GTM
(Graduate Texts in Math.), LNM (Lecture Notes in Math.), Grundl.MW (Grundlehren
Math.Wiss.).

References

[1] S. A. Amitsur, *Rational identities and application to algebra and geometry*, J. Alg. 3 (1966), pp. 304–359.

[2] H. Appelgate, M. Barr, J. Beck et al., *Seminar on triples and categorical homology theory*, ETH 1966/67, ed. B. Eckmann, LNM 80, Springer 1969.

[3] P. Ara, M. Martin, *Local multipliers of C^*-algebras*, Springer, London 2003.

[4] M. Artin, *Noncommutative deformations of commutative rings*, preprint 1994.

[5] M. Artin, J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. 109 (1994), no. 2, pp. 228–287.

[6] K. Asano, *Über die Quotientenbildung von Schiefringen*, (German) J. Math. Soc. Japan 1, (1949), pp. 73–78.

[7] P. Balmer, *Presheaves of triangulated categories and reconstruction of schemes*, Math. Ann. 324, no. 3 (2002), pp. 557-580.

[8] M. Barr, *Composite triples and derived functors*, in [2], pp. 336–356.

[9] M. Barr, C. Wells, *Toposes, triples and theories*, Grundl.MW 178, Springer 1985.

[10] J. Beck, *Distributive laws*, in [2], pp. 119–140.

[11] J. M. Beck, *Triples, algebras and cohomology*, Ph.D. Thesis, Columbia Univ. 1963.

[12] Kai Behrend, *Differential graded schemes I: perfect resolving algebras*, `math.AG/0212225`; *II: The 2-category of dg schemes*, `math.AG/0212226`.

[13] S. Beilinson, V. Drinfeld, *Chiral algebras*, AMS Colloq. Publ. 51 (2004) 352 pp.

[14] D. Ben-Zvi, T. Nevins, *From solitons to many-body systems*, `math.AG/0310490`

[15] G. M. Bergman, *The diamond lemma for ring theory*, Adv.M. 29 (1978), 178–219.

[16] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, LNM 1578 (1994).

[17] L. A. Bokut', P. S. Kolesnikov, *Gröbner-Shirshov bases: from inception to the present time*, Zap. Nauchn. Sem. POMI 272 (2000), Vopr. Teor. Predst. Algebr i Grupp. 7, 26–67, 345; transl. in J. Math. Sci. (N. Y.) 116 (2003), no. 1, 2894–2916.

[18] A. I. Bondal, M. M. Kapranov, *Representable functors, Serre functors, and reconstructions* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337; English translation in Math. USSR-Izv. 35 (1990), no. 3, pp. 519–541.

[19] A. Bondal, D. Orlov, *Derived categories of coherent sheaves*, `math.AG/0206295`.

[20] A. Bondal, M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3 (2003), no. 1, 1–36, 258; `math.AG/0204218`.

[21] F. Borceux, *Handbook of categorical algebra*, 3 vols. Enc. of Math. and its Appl. 50, Cambridge Univ. Press, 1994.

[22] N. Bourbaki, Éléments de mathématique: *Algèbre commutative, II*, 1964-9 Hermann, Paris, (Engl. transl. *Commutative Algebra* II, Addison Wesley 1972).

[23] I. Bucur, A. Deleanu, *Introduction to the theory of categories and functors*, Pure Appl. Math. XIX, Wiley 1968 (and Russian annotated transl. Mir, 1972).

[24] T. Brzeziński, S. Majid, *Coalgebra bundles*, Comm. M.Phys. 191 (1998), 2, 467–492.

[25] S. Caenepeel, G. Militaru, S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, LNM 1787, 2002.

[26] P. Cartier, *A mad day's work: from Grothendieck to Connes and Kontsevich – the evolution of concepts of space and symmetry*, Bull. AMS 38, No.4, pp. 389–408, 2001.

[27] P. M. Cohn, *Free rings and their relations*, Academic Press 1971.

[28] P. M. Cohn, *Inversive localization in Noetherian rings*, Comm. Pure. Appl. Math. 26 (1973), pp. 679–691.

[29] P. M. Cohn, *Algebra*, Vol. 3, 2nd ed. John Wiley and Sons, Chichester, 1991.

[30] A. Connes, *Noncommutative geometry*, Academic Press, New York 1994.

[31] M. Demazure, P. Gabriel, *Groupes algébriques I*, 1970 (French), *Introduction to algebraic geometry and algebraic groups* (Engl.transl. Ch.1-3), N. Holland 1980.

[32] J. Dixmier, *Enveloping algebras*, North Holland (1977) (enhanced version AMS 1997).

[33] M. Douglas, N. Nekrasov, *Noncommutative field theory*, Rev. Modern Phys. 73 (2001), no. 4, 977–1029.

[34] V. Drinfeld, *DG quotients of DG categories*, `math.KT/0210114`.

[35] C. Ehresmann, *Catégories et structures* (French), Paris, Dunod 1965, xvii+358 pp.

[36] S. Eilenberg, *Abstract description of some basic functors*, J. Indian Math. Soc. (N.S.) 24 (1960), 231–234.

[37] S. Eilenberg, J. C. Moore, *Adjoint functors and triples*, Illinois J. Math. 9 (1965) pp. 381–398.

[38] V. P. Elizarov, *Kol'ca častnyh* (Russian), Alg. i logika 8 (4) 1969, pp. 381–424; *K obščej teorii kolec častnyh*, Sib. Mat. Ž. 11 (3), 1970 pp. 526–546; *Sil'nye prekručenija i sil'nye fil'try, moduli i kol'ca častnyh*, Sib. Mat. Ž. 14 (3), 1973, pp. 549–559.

[39] C. Faith, *Algebra I,II*, Grundleheren Math. Wiss. 190, 191, Springer 1973,1976,1981.

[40] C. Faith, *Rings and things and a fine array of twentieth century associative algebra*, Math. Surveys & Monographs 65, AMS 1999.

[41] O. Gabber, L. Ramero, *Almost ring theory*, LNM 1800, Springer 2003, vi+307 pp.

[42] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448;

[43] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Springer 1967.

[44] S. I. Gelfand, Yu. I. Manin, *Homological algebra* (Russian), Cur. problems in math.. Fund. dir. 38, 5–240, Itogi Nauki i Tehniki, AN SSSR, VINITI 1989. (Engl. transl. Algebra, V, Encyclopaedia Math. Sci., 38, Springer, Berlin, 1994).

[45] S. I. Gelfand, Yu. I. Manin, *Methods of homological algebra*, Springer 1996; translated from the Russian original, Nauka 1988.

[46] I. M. Gel'fand, V. S. Retakh, *Determinants of matrices over noncommutative rings*, Fun. Anal. Appl. vol 25 (1991), no. 2, pp. 91–102. transl. 21 (1991), pp. 51–58.

[47] I. M. Gel'fand, V. S. Retakh, *A theory of noncommutative determinants and characteristic functions of graphs*, Fun. Anal. Appl. vol. 26 (1992), no.4, pp. 231–246.

[48] I. M. Gel'fand, V. S. Retakh, *Quasideterminants I*, Selecta Mathematica, New Series 3 (1997) no.4, pp. 517–546; `q-alg/9705026`.

[49] I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, *Quasideterminants*, `arXiv:math.QA/0208146`.

[50] V. N. Gerasimov, *Localization in assoc. rings*, Sib. M. Zh. 23-6 (1982) 36–54, 205; *Inverting homomorphisms of rings*, Algebra i Logika 18-6 (1979) 648–663, 754.

[51] J. Giraud, *Cohomologie non abélienne*, Grundl. Math. Wiss. 179, Springer 1971.

[52] J. S. Golan, J. Raynaud, F. van Oystaeyen, *Sheaves over the spectra of certain noncommutative rings*, Comm. Alg. 4(5), (1976), pp. 491–502.

[53] O. Goldman, *Rings and modules of quotients*, J. Algebra 13 (1969), pp. 10–47.

[54] K. R. Goodearl, R. B. Warfield Jr., *An introduction to noncommutative Noetherian rings*, LMS Student Texts, Vol. 16, Cambridge Univ. Press 1989.

[55] J. M. Gracia-Bondía, J. Várilly, H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehbúcher, 2001, xviii+685 pp.

[56] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale de schémas et de morphismes de schémas*, Publ. Math. IHES 32 (1967).

[57] A. Grothendieck et al., *Revêtements étales et groupe fondamental*, Séminaire de Géometrie Algébrique du Bois Marie 1960–1961 (SGA 1), LNM 224, Springer (1971), retyped as: `arXiv:math.AG/0206203`.

[58] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J.(2) 9 (1957) pp. 119–221.

[59] R. Hartshorne, *Algebraic geometry*, Springer, GTM 52 (1977) xvi+496 pp.

[60] R. Hartshorne, *Residues and duality* (& appendix by P. Deligne), Springer 1966.

[61] T. J. Hodges, S. P. Smith, *Sheaves of noncommutative algebras and the Beilinson-Bernstein equivalence of categories*, Proc. AMS, 92, N. 3, 1985.

[62] P. Jara, A. Verschoren, C. Vidal, *Localization and sheaves: a relative point of view*, Pitman Res. Notes in Math. 339, Longman 1995.

[63] P. T. Johnstone, *Topos theory*, LMS Monographs 10, Acad. Press 1977; *Sketches of an elephant: a topos theory compendium*, 3 vols, Oxford Univ. Press 2002.

[64] M. Kapranov, *Noncommutative geometry based on commutator expansions*, J. Reine Angew. Math. 505 (1998), pp. 73–118; `arXiv:math.AG/9802041`.

[65] B. Keller, *Introduction to A-infinity algebras and modules*, Homology, Homotopy and Applications 3 (2001), no. 1, 1–35 (electronic).

[66] A. Kapustin, A. Kuznetsov, D. Orlov, *Noncommutative instantons and twistor transform*, Comm. Math. Phys. 221 (2001), pp. 385–432; `arXiv:hep-th/0002193`.

[67] M. Kontsevich, *Deformation quantization for algebraic varieties*, `math.QA/0106006`.

[68] M. Kontsevich, A. L. Rosenberg, *Noncommutative smooth spaces*, The Gelfand Math. Seminars, 1996–1999, Birkhäuser 2000, pp. 85–108; `arXiv:math.AG/9812158`.

[69] M. Kontsevich, A. L. Rosenberg, *Noncommutative spaces*, MPI 2004 - 35; *Noncommutative spaces and flat descent*, MPI 2004 - 36; *Noncommutative stacks*, MPI 2004 - 37; preprints, Bonn 2004.

[70] M. Kontsevich, Y. Soibelman, *Homological mirror symmetry and torus fibrations*, in: Symplectic geometry and mirror symmetry (Seoul, 2000), pp. 203–263, World Sci. Publ. (2001) `math.SG/0011041`.

[71] D. KROB, B. LECLERC, *Minor identities for quasi-determinants and quantum determinants*, Comm.Math.Phys. 169 (1995), pp. 1–23.

[72] J. LAMBEK, *Torsion theories, additive semantics and rings of quotients*, LNM 177.

[73] N.P. LANDSMAN, *Bicategories of operator algebras and Poisson manifolds*, Math. physics in math. & physics (Siena, 2000), Fields Inst. Comm. 30, 271–286, AMS 2001.

[74] N. P. LANDSMAN, *Math. topics between class. & quantum mechanics*, Springer 1998.

[75] O. A. LAUDAL, *Noncommutative algebraic geometry*, preprint MPI 2000-115.

[76] D. A. LEITES, *Introduction to the theory of supermanifolds* (Russian), Usp. Mat. Nauk. 35 (1980), pp. 3–57.

[77] W. T. LOWEN, M. VAN DEN BERGH, *Def. theory of ab. categories*, `math.CT/0405226`; *Hochschild cohomology of abelian categories and ringed spaces*, `math.KT/0405227`.

[78] V. A. LUNTS, *Deformations of quasicoherent sheaves of algebras*, J. Alg. 259 (2003), pp. 59-86; `arXiv:math.AG/0105006`.

[79] V. A. LUNTS, A. L. ROSENBERG, *Differential calculus in noncommutative algebraic geometry*, Max Planck Institute Bonn preprints:

I. D-calculus on noncommutative rings, MPI 96-53;

II. D-calculus in the braided case. The localization of quantized enveloping algebras, MPI 96-76, Bonn 1996.

[80] V. A. LUNTS, A. L. ROSENBERG, *Differential operators on noncommutative rings*, Sel. Math. N. S. 3 (1997), pp. 335–359.

[81] V. A. LUNTS, A. L. ROSENBERG, *Localization for quantum groups*, Selecta Math. (N.S.) 5 (1999), no. 1, pp. 123–159.

[82] V. A. LUNTS, A. L. ROSENBERG, *Localization for quantum groups II*, manuscript.

[83] G. LUSZTIG, *Introduction to quantum groups*, Progr. in Math. 110, Birkhäuser, 1993.

[84] S. MAC LANE, *Categories for the working mathematician*, GTM 5, Springer 1971.

[85] S. MAC LANE, I. MOERDIJK, *Sheaves in geometry and logic*, Springer 1992.

[86] P. MALCOLMSON, *Construction of universal matrix localizations*, pp. 117–131, Advances in Non-comm. Ring theory. Proc. 12th G. H. Hudson Symp. Plattsburgh, 1981. LNM 951, Springer 1982.

[87] YU. I. MANIN, *Quantum groups & non-commutative geometry*, CRM, Montreal 1988.

[88] YU. I. MANIN, *Moduli, motives, mirrors*, pp. 53–73, Proc. 3rd Eur. Cong. Math. Barcelona July 10–14, 2000, Prog. Math. 201; `math.AG/0005144`.

[89] YU. I. MANIN, *Topics in noncommutative geometry*, Princeton Univ. Press 1991.

[90] H. FAUSK, PO HU, J. P. MAY, *Isomorphisms between left and right adjoints*, Theory Appl. Categ. 11 (2003), pp. 107–131 (electronic).

[91] J. C. MCCONNELL, J. C. ROBSON, *Noncommutative noetherian rings*, 1987.

[92] F. MOREL, *An introduction to $\mathbb{A}^1$-homotopy Theory*, Cont. dev. in alg. K-theory, M. Karoubi, A. O. Kuku, C. Pedrini eds. ICTP lec. notes series 15 (2004).

[93] D. C. MURDOCH, F. VAN OYSTAEYEN, *Noncommutative localization and sheaves*, J. Alg. 38 (1975), pp. 500–515.

[94] S. MONTGOMERY, H.-J. SCHNEIDER, *Hopf crossed products, rings of quotients and prime ideals*, Adv. Math. 112 (1995), pp. 1–55.

[95] J. NDIRAHISHA, F. VAN OYSTAEYEN, *Grothendieck representations of categories and canonical noncommutative topologies*, K-theory 25 (2002), pp. 355-371.

[96] A. NEEMAN, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. AMS 9 (1996), pp. 205–236.

[97] A. NEEMAN, A. RANICKI, *Noncommutative localization and chain complexes I. Algebraic K- and L-theory*, `math.RA/0109118`, 75 pp.

[98] O. ORE, *Linear equations in noncomm. fields*, Ann. Math. 32 (1931), pp. 463–477.

[99] D. O. ORLOV, *Quasi-coherent sheaves in commutative and non-commutative geometry*, Izvestiya: Math. 67:3 (2003), pp. 119–138.

[100] D. O. ORLOV, *Derived categories of coherent sheaves and equivalences between them*, (Russian) Uspehi Mat. Nauk 58 (2003), no. 3, pp. 89–172.

[101] C. J. PAPPACENA, *Frobenius bimodules between noncommutative spaces*, `math.QA/0304386`.

[102] A. POLISHCHUK, M. ROTHSTEIN, *Fourier transform for D-algebras*, `math.AG/9901009`.

[103] N. POPESCU, *Abelian categories with applications to rings and modules*, London Math. Soc. Monographs 3, Academic Press (1973).

[104] L. POPESCU, N. POPESCU, *Theory of categories*, Bucuresti & Alphen: Editura Academiei & Sijthoff & Noordhoff International Publishers, 1979.

[105] T. PORTER, *$\mathcal{S}$-categories, $\mathcal{S}$-groupoids, Segal categories and quasicategories*, `math.AT/0401274`.

[106] N. RESHETIKHIN, A. VORONOV, A. WEINSTEIN, *Semiquantum geometry*, Algebraic geometry 5, J. Math. Sci. 82 (1996), no. 1, pp. 3255–3267; `q-alg/9606007`.

[107] A. L. ROSENBERG, *Noncomm. local algebra*, Geom. Fun. Anal. 4 (5), 1994, 545–585.

[108] A. L. ROSENBERG, *The spectrum of abelian categories and reconstruction of schemes*, in Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels, 1996), pp. 257–274, Lec. Notes Pure Appl. Math. 197, Dekker, NY 1998. ('96 version: MPI-1996-108).

[109] A. L. ROSENBERG, *Noncommutative schemes*, Comp. Math. 112 (1998), pp. 93–125.

[110] A. L. ROSENBERG, *Noncommutative spaces and schemes*, MPI-1999-84, Bonn 1999.

[111] A. L. ROSENBERG, *Spectra related with localizations*, MPI-2003-112, Bonn 2003.

[112] A. L. ROSENBERG, *Non-commutative affine semischemes and schemes*, Seminar on supermanifolds No. 26, edited by D. Leites, Dept. of Math., Univ. of Stockholm 1988.

[113] A. L. ROSENBERG, *Noncommutative algebraic geometry and representations of quantized algebras*, MAIA 330, Kluwer 1995.

[114] A. L. ROSENBERG, *Spectra of noncommutative spaces*, MPI-2003-110, Bonn 2003.

[115] A. L. ROSENBERG, *Underlying spectra of noncommutative schemes*, MPI-2003-111.

[116] L. ROWEN, *Ring theory*, 2 vols, Acad. Press (Pure & Applied Math. 127,128), 1988.

[117] A. C. DA SILVA, A. WEINSTEIN, *Geometric models for noncommutative algebra*, UC Berkeley, AMS, 1999, 184 pp.

[118] Z. Škoda, *Coherent states for Hopf algebras*, `math.QA/0303357`.

[119] Z. Škoda, *Distributive laws for actions of monoidal categories*, `math.CT/0406310`.

[120] Z. Škoda, *Globalizing Hopf-Galois extensions*, preprint;
Quantum bundles using coactions and localization, in preparation.

[121] Z. Škoda, *Localized coinvariants I,II*, preprints.

[122] Z. Škoda, *Localizations for construction of quantum coset spaces*, in "Noncommutative geometry and quantum groups", P. M. Hajac, W. Pusz eds. Banach Center Publications vol.61, pp. 265–298, Warszawa 2003; `arXiv:math.QA/0301090`.

[123] Z. Škoda, *Universal noncommutative flag variety*, preliminary version.

[124] P. S. Smith, *Subspaces of non-commutative spaces*, `math.QA/0104033`;
Integral non-commutative spaces, `math.QA/0104046`;
Maps between non-commutative spaces, `math.QA/0208134`.

[125] P. S. Smith, *Non-commutative algebraic geometry*, preprint notes, U. Wash. 2000.

[126] Y. Soibelman, *Quantum tori, mirror symmetry and deformation theory*, Lett. Math. Phys. 56 (2001), no. 2, 99–125.`math.QA/0011162`.
with P. Bressler, *Mirror symmetry and deformation quantization*, `hep-th/0202128`.
with V. Vologodsky, *Non-commutative compactifications and elliptic curves*, `math.AG/0205117`.

[127] J. T. Stafford, *Noncommutative projective geometry*, Proceedings of the ICM, Beijing 2002, vol. 2, pp. 93–104; `math.RA/0304210`.

[128] J. T. Stafford, T. A. Nevins, *Sklyanin algebras and Hilbert schemes of points*, `math.AG/0310045`.

[129] J. T. Stafford, M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. AMS 38 (2001), pp.171–216.

[130] Bo Stenström, *Rings of quotients*, Grundlehren Math. Wiss. 217, Springer 1975.

[131] R. Street, *The formal theory of monads*, J. Pure & Appl. Alg. 2 (1972), 149–168; & part II (with S. Lack), J. Pure Appl. Algebra 175 (2002), no. 1-3, 243–265.

[132] B. Toen, *Homotopical and higher categorical structures in algebraic geometry*, `math.AG/0312262`; *Affine stacks* (Champs affines), `math.AG/0012219`.

[133] B. Toen, G. Vezzosi, *Algebraic geometry over model categories* etc. `math.AG/0110109,0207028,0210407,0212330`, `math.AT/0309145`.

[134] R. W. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, vol. III, Progr. Math. 88, Birkhäuser (1990), pp. 247–435.

[135] M. Van den Bergh, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. 154, 2001.

[136] A. van den Essen, *Alg. microlocalization and modules with regular singularities over filtered rings*, Handbook of algebra, v.1, ed. M. Hazewinkel, 813–840, N. Holland, 1996.

[137] A. Verschoren, *Sheaves and localization*, J. Algebra 182 (1996), no. 2, pp. 341–346.

[138] F. van Oystaeyen, *Algebraic geometry for associative algebras*, Marcel Dekker 2000.

[139] F. van Oystaeyen, A. Verschoren, *Reflectors and localization, application to sheaf theory*, Lec. Notes in Pure. Appl. Math. 41, M. Dekker 1979.

[140] F. van Oystaeyen, A. Verschoren, *Noncommutative algebraic geometry. An introduction.* LNM 887, Springer 1981.

[141] F. van Oystaeyen, L. Willaert, *Grothendieck topology, coherent sheaves and Serre's theorem for schematic algebras*, J. Pure Appl. Alg. 104 (1995), pp. 109–122. *Cohomology of schematic algebras*, J. Alg. 185 (1996), pp. 74–84.

[142] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory.*

[143] C. E. Watts, *Intrinsic characterization of some additive functors*, Proc. AMS 11 (1960), pp. 1–8.

[144] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press 1994.

[145] R. Wisbauer, *Module and comodule categories – a survey*, Proc. of the Math. Conf. Birzeit/Nablus, 1998, World Scientific (2000), pp. 277–304.

[146] H. Wolff, *$\mathcal{V}$-localizations and $\mathcal{V}$-monads*, J. Alg. 24 (1973), pp. 405–438.

[147] A. Yekutieli, *On deformation quantization in algebraic geometry*, `math.AG/0310399`.

[148] A. Yekutieli, J. L. Zhang, *Dualizing complexes and perverse sheaves on noncommutative ringed schemes*, `math.QA/0211309`.

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